

# RATIONAL HOMOTOPY THEORY OF AUTOMORPHISMS OF HIGHLY CONNECTED MANIFOLDS

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**ABSTRACT.** We study the rational homotopy types of classifying spaces of automorphism groups of  $2d$ -dimensional  $(d-1)$ -connected manifolds ( $d \geq 3$ ). We prove that the rational homology groups of the homotopy automorphisms and the block diffeomorphisms of the manifold  $\#^g S^d \times S^d$  relative to a disk stabilize as  $g$  increases. Via a theorem of Kontsevich, we obtain the striking result that the stable rational cohomology of the homotopy automorphisms comprises all unstable rational homology groups of all outer automorphism groups of free groups.

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## 1. INTRODUCTION

The manifold

$$M_g = \#^g S^d \times S^d$$

is an example of a *highly connected manifold*: it is  $2d$ -dimensional and  $(d-1)$ -connected. Let  $M_{g,1}$  denote the result of removing an open embedded  $2d$ -disk from  $M_g$ , and let

$$X_{g,1} = B \operatorname{aut}_{\partial}(M_{g,1})$$

denote the classifying space of the topological monoid of homotopy self-equivalences of  $M_{g,1}$  that restrict to the identity on the boundary.

For  $d=1$ , the manifold  $M_{g,1}$  is an orientable genus  $g$  surface with one boundary component, and the classifying space  $X_{g,1}$  has the same rational cohomology as the moduli space  $\mathcal{M}_{g,1}$  of Riemann surfaces. By Harer's work [26], the cohomology  $H^k(X_{g,1}; \mathbb{Q})$  stabilizes as  $g$  increases, and Mumford's conjecture (settled in the positive [34]) describes the limit as  $g$  tends to infinity: it is a polynomial algebra in the Morita-Miller-Mumford classes,

$$H^*(X_{\infty,1}; \mathbb{Q}) \cong \mathbb{Q}[\kappa_1, \kappa_2, \dots], \quad |\kappa_i| = 2i. \quad (d=1).$$

In this paper we study the classifying space  $X_{g,1}$  for  $d \geq 3$  from the point of view of rational homotopy theory. Our first main result is an analog of Harer's homological stability theorem.

**Theorem 1.1.** *Let  $d \geq 3$ . The stabilization map*

$$H_k(B \operatorname{aut}_{\partial}(M_{g,1}); \mathbb{Q}) \rightarrow H_k(B \operatorname{aut}_{\partial}(M_{g+1,1}); \mathbb{Q}),$$

*is an isomorphism for  $g > 2k+4$  and surjective for  $g = 2k+4$ .*

The cohomology also stabilizes and we may consider the limit

$$H^*(B \operatorname{aut}_{\partial}(M_{\infty,1}); \mathbb{Q}) = \lim_g H^*(B \operatorname{aut}_{\partial}(M_{g,1}); \mathbb{Q}).$$

Our second main result is a calculation of the stable cohomology ring in terms of the stable cohomology of arithmetic groups and Lie algebra cohomology.

Let  $\Gamma_g$  denote the symplectic group  $\operatorname{Sp}_{2g}(\mathbb{Z})$  if  $d$  is odd and the orthogonal group  $O_{g,g}(\mathbb{Z})$  if  $d$  is even. Let  $\mathfrak{g}_g$  denote the graded Lie algebra of positive degree derivations on the free graded Lie algebra over  $\mathbb{Q}$  on generators  $\alpha_1, \beta_1, \dots, \alpha_g, \beta_g$  of degree  $d-1$  that annihilate the element

$$\omega_g = [\alpha_1, \beta_1] + \dots + [\alpha_g, \beta_g].$$

There is a natural action of  $\Gamma_g$  on  $\mathfrak{g}_g$ . There is an inclusion  $\mathfrak{g}_g \subset \mathfrak{g}_{g+1}$  given by extending a derivation by zero on  $\alpha_{g+1}$  and  $\beta_{g+1}$ , and there is an evident inclusion  $\Gamma_g \subset \Gamma_{g+1}$ . We let  $\mathfrak{g}_{\infty} = \cup_g \mathfrak{g}_g$  and  $\Gamma_{\infty} = \cup_g \Gamma_g$ .

**Theorem 1.2.** *For  $d \geq 3$ , there is an isomorphism of graded algebras*

$$H^*(B \operatorname{aut}_{\partial}(M_{\infty,1}); \mathbb{Q}) \cong H^*(\Gamma_{\infty}; \mathbb{Q}) \otimes H_{CE}^*(\mathfrak{g}_{\infty})^{\Gamma_{\infty}}.$$

Borel's calculation [12] allows us to identify the left tensor factor with a polynomial algebra,

$$H^*(\Gamma_{\infty}; \mathbb{Q}) \cong \begin{cases} \mathbb{Q}[x_1, x_2, \dots], & |x_i| = 4i-2, \quad d \text{ odd}, \\ \mathbb{Q}[x_1, x_2, \dots], & |x_i| = 4i, \quad d \text{ even}. \end{cases}$$

Serendipitously, the Lie algebra  $\mathfrak{g}_g$ , for  $d$  odd, has appeared before in the work of Kontsevich [28, 29]. By using Kontsevich's results, we arrive at the striking conclusion that the second factor is comprised of the rational homology groups of all outer automorphism groups of free groups. More precisely, we have the following.

**Theorem 1.3.** *For  $d \geq 3$  odd, the stable cohomology ring*

$$H^*(B \operatorname{aut}_{\partial}(M_{\infty,1}); \mathbb{Q})$$

*is a free graded commutative algebra on*

*‘Borel classes’  $x_1, x_2, \dots$ , of degree  $|x_i| = 4i - 2$ , and*

*a class  $\lambda_c$  for every  $c \in \cup_{n,k} B_{n,k}$ , of degree  $|\lambda_c| = 2nd - k$ .*

*Here  $B_{n,k}$  is a basis for the vector space  $H_k(\operatorname{Out} F_{n+1}; \mathbb{Q})$ , and  $\operatorname{Out} F_{n+1}$  denotes the outer automorphism group of the free group on  $n + 1$  generators.*

When  $d$  is even, Kontsevich’s theorem does not yield any information. We plan to address the calculation of the stable cohomology ring for  $d$  even in a sequel paper.

Our result should be compared to the calculation of the stable cohomology ring of the diffeomorphism group  $\operatorname{Diff}_{\partial}(M_{g,1})$  due to Galatius and Randal-Williams [24] (recalled in Section §6.2). The rational homology of  $\operatorname{Out} F_n$  is not fully understood, but it has been the subject of a lot of recent research, see, e.g., [19, 44], and it is possible to draw some immediate interesting conclusions.

**Corollary 1.4.** *For  $d \geq 3$  odd, the map in cohomology*

$$(1) \quad H^*(B \operatorname{aut}_{\partial}(M_{\infty,1}); \mathbb{Q}) \rightarrow H^*(B \operatorname{Diff}_{\partial}(M_{\infty,1}); \mathbb{Q})$$

*is non-trivial, but neither surjective nor injective.*

This is in sharp contrast with what happens for surfaces. For  $d = 1$ , the manifold  $M_{g,1}$  is an orientable genus  $g$  surface with one boundary component, and the inclusion of  $\operatorname{Diff}_{\partial}(M_{g,1})$  into  $\operatorname{aut}(M_{g,1})$  is a homotopy equivalence if  $g > 1$ . In particular, the map (1) is an isomorphism for  $d = 1$ .

It would be interesting to get a better understanding of the map (1) for  $d \geq 3$ . It factors through the cohomology ring of the block diffeomorphism group  $\widetilde{\operatorname{Diff}}_{\partial}(M_{g,1})$  (see §5.5), so it would be desirable to get a description of this ring. To this end we prove the following theorem.

**Theorem 1.5.** *Let  $d \geq 3$ . The map*

$$H_k(\widetilde{B \operatorname{Diff}}(M_{g,1}); \mathbb{Q}) \rightarrow H_k(\widetilde{B \operatorname{Diff}}(M_{g+1,1}); \mathbb{Q}),$$

*is an isomorphism for  $g > 2k + 4$  and surjective for  $g = 2k + 4$ .*

This greatly improves the range obtained in [3]. A simple description of the stable cohomology ring of the block diffeomorphism group would be highly interesting. We hope to return to this in a sequel paper.

The proofs of the above results take their starting point in the following general description of the rational homotopy types of the classifying spaces of homotopy automorphisms of highly connected manifolds.

**Theorem 1.6.** *Let  $M$  be a closed  $(d - 1)$ -connected  $2d$ -dimensional manifold and let  $N$  denote the result of removing an open  $2d$ -disk from  $M$ . Let  $X$  denote either of the classifying spaces*

$$B \operatorname{aut}(M), \quad B \operatorname{aut}_*(M), \quad \text{or} \quad B \operatorname{aut}_{\partial}(N),$$

*and  $\tilde{X}$  the simply connected cover of  $X$ . Let  $H = H_d(M; \mathbb{Z})$  with intersection form  $\mu$  and quadratic refinement  $Jq$  (see §4.1). If  $d \geq 3$  and  $\operatorname{rank} H > 2$ , then*

- (1) *The fundamental group  $\pi_1(X)$  maps surjectively, with finite kernel, onto the automorphism group  $\operatorname{Aut}(H, \mu, Jq)$ .*
- (2) *Quillen’s dg Lie algebra  $\lambda(\tilde{X})$  is formal.*

- (3) *The rational homotopy Lie algebra  $\pi_*^{\mathbb{Q}}(\tilde{X}) = \pi_{*+1}(\tilde{X}) \otimes \mathbb{Q}$ , with the Whitehead product, is isomorphic to*

$$\text{OutDer}^+ \pi_*^{\mathbb{Q}}(M), \quad \text{Der}^+ \pi_*^{\mathbb{Q}}(M), \quad \text{or} \quad \text{Der}_{\omega}^+ \pi_*^{\mathbb{Q}}(N);$$

*the graded Lie algebra of positive degree outer derivations, derivations, or derivations annihilating  $\omega$ , respectively. The graded Lie algebra  $\pi_*^{\mathbb{Q}}(N)$  is free on rank  $H$  generators of degree  $d-1$ , and  $\pi_*^{\mathbb{Q}}(M)$  is isomorphic to the quotient graded Lie algebra  $\pi_*^{\mathbb{Q}}(N)/(\omega)$ , where  $\omega \in \pi_{2d-1}(N)$  is the homotopy class of the attaching map for the top cell in  $M$ .*

We remark that Theorem 1.6 may be viewed as an ‘infinitesimal’ version of the Dehn-Nielsen-Baer theorem (see, e.g., [22, Chapter 8]).

## 2. QUILLEN’S RATIONAL HOMOTOPY THEORY

In this section we will briefly review Quillen’s rational homotopy theory [45] and set up a spectral sequence for calculating the rational homology of a simply connected space from its rational homotopy groups. The existence of this spectral sequence was pointed out by Quillen [45, §6.9], but we need a version that incorporates group actions that are not necessarily base-point preserving, so we need to review the construction in detail.

**2.1. Quillen’s dg Lie algebra.** The Whitehead products on the homotopy groups of a simply connected based topological space  $X$ ,

$$\pi_{p+1}(X) \times \pi_{q+1}(X) \rightarrow \pi_{p+q+1}(X),$$

endow the rational homotopy groups,

$$\pi_*^{\mathbb{Q}}(X) = \pi_{*+1}(X) \otimes \mathbb{Q},$$

with the structure of a graded Lie algebra. Rationally homotopy equivalent spaces have isomorphic Lie algebras, but  $\pi_*^{\mathbb{Q}}(X)$  is not a complete invariant; two spaces may have isomorphic Lie algebras without being rationally homotopy equivalent, as witnessed for instance by  $\mathbb{CP}^2$  and  $K(\mathbb{Z}, 2) \times K(\mathbb{Z}, 5)$ .

Quillen [45] constructed a functor  $\lambda$  from the category of simply connected based topological spaces to the category of dg Lie algebras and established a natural isomorphism of graded Lie algebras

$$(2) \quad H_*(\lambda(X)) \cong \pi_*^{\mathbb{Q}}(X).$$

The quasi-isomorphism type of  $\lambda(X)$  is a finer invariant than the isomorphism type of  $\pi_*^{\mathbb{Q}}(X)$ . The main result of Quillen’s theory is that it is a complete invariant: two simply connected spaces  $X$  and  $Y$  have the same rational homotopy type if and only if the dg Lie algebras  $\lambda(X)$  and  $\lambda(Y)$  are quasi-isomorphic. Here, we say that two dg Lie algebras are quasi-isomorphic if they are isomorphic in the homotopy category of dg Lie algebras. Concretely, this means that there exists a zig-zag of quasi-isomorphisms that connects them.

**2.2. The Quillen spectral sequence.** Let  $L$  be a dg Lie algebra. The Chevalley-Eilenberg complex of  $L$  is the chain complex

$$C_*^{CE}(L) = (\Lambda sL, \delta).$$

Here  $\Lambda sL$  denotes the free graded commutative algebra on the suspension of  $L$ . Elements of  $sL$  are denoted  $sx$ , where  $x \in L$ , with  $|sx| = |x| + 1$ . The differential  $\delta = \delta_0 + \delta_1$  is defined by the following formulas

$$\delta_0(sx_1 \wedge \dots \wedge sx_n) = \sum_{i=1}^n (-1)^{1+\epsilon_i} sx_1 \wedge \dots \wedge \delta x_i \wedge \dots \wedge sx_n,$$

$$\delta_1(sx_1 \wedge \dots \wedge sx_n) = \sum_{i < j} (-1)^{|sx_i| + \eta_{ij}} s[x_i, x_j] \wedge sx_1 \wedge \dots \widehat{sx_i} \dots \widehat{sx_j} \dots \wedge sx_n,$$

where

$$\epsilon_i = |sx_1| + \dots + |sx_{i-1}|,$$

and the sign  $(-1)^{\eta_{ij}}$  is determined by graded commutativity;

$$sx_1 \wedge \dots \wedge sx_n = (-1)^{\eta_{ij}} sx_i \wedge sx_j \wedge sx_1 \dots \widehat{sx_i} \dots \widehat{sx_j} \dots \wedge sx_n.$$

We let  $H_*^{CE}(L)$  denote the homology of this chain complex.

If the differential of  $L$  is trivial, then there is a decomposition of the Chevalley-Eilenberg homology as

$$H_n^{CE}(L) = \bigoplus_{p+q=n} H_{p,q}^{CE}(L),$$

where  $H_{p,q}^{CE}(L)$  is the homology in word-length  $p$  and total degree  $p+q$ .

$$\dots \longrightarrow (\Lambda^{p+1} sL)_q \xrightarrow{\delta_1} (\Lambda^p sL)_q \xrightarrow{\delta_1} (\Lambda^{p-1} sL)_q \longrightarrow \dots$$

For arbitrary  $L$ , we may filter the Chevalley-Eilenberg complex by word-length;

$$F_p = \Lambda^{\leq p} sL.$$

The associated spectral sequence has

$$(3) \quad E_{p,q}^2(L) = H_{p,q}^{CE}(H_*(L)) \Rightarrow H_{p+q}^{CE}(L)$$

If  $L$  is positively graded the filtration is finite in each degree, which ensures convergence of the spectral sequence.

There is a coproduct on  $\Lambda sL$ , called the shuffle coproduct, which is uniquely determined by the requirement that it makes  $\Lambda sL$  into a graded Hopf algebra with space of primitives  $sL$ . The differential  $\delta$  is a coderivation with respect to the shuffle coproduct, making  $C_*^{CE}(L)$  into a dg coalgebra, and (3) is a spectral sequence of coalgebras.

We will now interpret the above for the dg Lie algebra  $\lambda(X)$ . A fundamental property of Quillen's functor is the existence of a natural isomorphism of graded coalgebras

$$(4) \quad H_*^{CE}(\lambda(X)) \cong H_*(X; \mathbb{Q}).$$

By (2) and (4) the spectral sequence of Quillen's dg Lie algebra  $\lambda(X)$  may be written as follows

$$(5) \quad E_{p,q}^2(X) = H_{p,q}^{CE}(\pi_*^{\mathbb{Q}}(X)) \Rightarrow H_{p+q}(X; \mathbb{Q}).$$

We will refer to this as the Quillen spectral sequence.

**Functoriality for unbased maps.** It is clear from the construction that the Quillen spectral sequence is natural for base-point preserving maps. But in fact the functoriality can be extended to unbased maps. The homotopy groups  $\pi_n(X) = [S^n, X]_*$  depend on the base-point of  $X$ , and are a priori only functorial for base-point preserving maps. However, if  $X$  is simply connected, the canonical map

$$\pi_n(X) \rightarrow [S^n, X]$$

is a bijection, and we may use this to extend  $\pi_*^{\mathbb{Q}}(X)$  to a functor defined on unbased simply connected spaces. Quillen's functor  $\lambda(X)$  can also be extended to unbased maps, but only up to homotopy.

Suppose that  $X$  and  $Y$  are simply connected spaces with base-points  $x_0$  and  $y_0$ . Given a not necessarily base-point preserving map  $f: X \rightarrow Y$ , we may choose a path  $\gamma$  from  $y_0$  to  $f(x_0)$ . Then we obtain based maps

$$(X, x_0) \xrightarrow{f} (Y, f(x_0)) \xleftarrow{ev_1} (Y^I, \gamma) \xrightarrow{ev_0} (Y, y_0).$$

The maps  $ev_i$  are weak homotopy equivalences, so the above may be interpreted as a morphism  $\bar{f}$  from  $(X, x_0)$  to  $(Y, y_0)$  in the homotopy category of based spaces. It is easily checked that  $\bar{f}$  only depends on the homotopy class of  $f$ , and that compositions are respected in the sense that  $\overline{gf} = \bar{g}\bar{f}$  as maps in the homotopy category.

We may apply Quillen's functor to get a diagram of dg Lie algebras

$$\lambda(X, x_0) \xrightarrow{f_*} \lambda(Y, f(x_0)) \xleftarrow{(ev_1)_*} \lambda(Y^I, \gamma) \xrightarrow{(ev_0)_*} \lambda(Y, y_0),$$

where the maps  $(ev_i)_*$  are quasi-isomorphisms. In homotopy, we obtain an induced morphism of graded Lie algebras

$$(ev_0)_*(ev_1)_*^{-1} f_*: \pi_*^{\mathbb{Q}}(X) \rightarrow \pi_*^{\mathbb{Q}}(Y).$$

Under the identification  $\pi_n(X) \cong [S^n, X]$ , this map agrees with  $f_*: [S^n, X] \rightarrow [S^n, Y]$ , because  $ev_0$  and  $ev_1$  are homotopic as unbased maps. Since the spectral sequence (3) is natural with respect to morphisms of dg Lie algebras, the above considerations imply the following.

**Proposition 2.1.** *Let  $X$  be a simply connected space. There is a spectral sequence of coalgebras*

$$E_{p,q}^2 = H_{p,q}^{CE}(\pi_*^{\mathbb{Q}}(X)) \Rightarrow H_{p+q}(X; \mathbb{Q}).$$

*The spectral sequence is natural with respect to unbased maps of simply connected spaces.*

In particular, if  $X$  has a not necessarily base-point preserving action of a group  $\pi$ , then the Quillen spectral sequence (5) is a spectral sequence of  $\pi$ -modules (from the  $E^1$ -page and on). An important special case is when  $X = \tilde{Y}$  is the universal cover of a path connected space  $Y$  and  $\pi$  is the group of deck transformations. By the above, we obtain a spectral sequence of coalgebras with a  $\pi$ -action,

$$E_{p,q}^2 = H_{p,q}^{CE}(\pi_*^{\mathbb{Q}}(\tilde{Y})) \Rightarrow H_{p+q}(\tilde{Y}; \mathbb{Q}).$$

It is an exercise in covering space theory to check that, under the standard identifications

$$\pi \cong \pi_1(Y), \quad \pi_n(\tilde{Y}) \cong \pi_n(Y), \quad n \geq 2,$$

the action of  $\pi$  on  $\pi_n(\tilde{Y})$  obtained as above corresponds to the usual action of  $\pi_1(Y)$  on the higher homotopy groups  $\pi_n(Y)$ .

**2.3. Formality and collapse of the Quillen spectral sequence.** The spectral sequence (3) is natural with respect to morphisms of dg Lie algebras. Evidently, a quasi-isomorphism induces an isomorphism from the  $E^1$ -page and on, so quasi-isomorphic dg Lie algebras have isomorphic spectral sequences. It is also evident that the spectral sequence of a dg Lie algebra with trivial differential collapses at the  $E^2$ -page. These simple observations have an interesting consequence. Namely, if the dg Lie algebra  $L$  is *formal*, meaning that it is quasi-isomorphic to its homology  $H_*(L)$  viewed as a dg Lie algebra with trivial differential, then the spectral sequence for  $L$  collapses at the  $E^2$ -page. Collapse of the spectral sequence is weaker than formality in general, although the difference is subtle.

**Definition 2.2.** Let us say that a group  $\pi$  is *rationally perfect* if  $H^1(\pi; V) = 0$  for every finite dimensional  $\mathbb{Q}$ -vector space  $V$  with an action of  $\pi$ .

**Proposition 2.3.** *Let  $\pi$  be a group acting on a simply connected space  $X$  with degree-wise finite dimensional rational cohomology groups. If  $\pi$  is rationally perfect*

and if Quillen's dg Lie algebra  $\lambda(X)$  is formal, then there is an isomorphism of graded  $\pi$ -modules

$$H_n(X; \mathbb{Q}) \cong \bigoplus_{p+q=n} H_{p,q}^{CE}(\pi_*^{\mathbb{Q}}(X)),$$

for every  $n$ .

*Proof.* If the rational cohomology groups of a simply connected space are finite dimensional, then so are the rational homotopy groups. It follows that the Quillen spectral sequence (5) is a spectral sequence of finite dimensional  $\pi$ -modules. Since  $\lambda(X)$  is formal, the Quillen spectral sequence collapses, and since  $\pi$  is rationally perfect, all extensions relating  $E_{*,*}^{\infty}$  and  $H_*(X; \mathbb{Q})$  are split.  $\square$

### 3. CLASSIFICATION OF FIBRATIONS

The purpose of this section is to review some fundamental results on the classification of fibrations in the categories of topological spaces and dg Lie algebras.

The classification of fibrations up to fiber homotopy equivalence was pioneered by Stasheff [48] and given a systematic treatment by May [38]. For a more recent modern approach, see [10]. The classification of fibrations for dg Lie algebras is implicit in the work of Sullivan [49] and in a widely circulated preprint of Schlessinger-Stasheff (recently made available [47]). A detailed account is given in Tanré's book [50]. There is also a more recent approach due to Lazarev [31], which uses the language of  $L_{\infty}$ -algebras.

**3.1. Fibrations of topological spaces.** Let  $X$  be a simply connected space of the homotopy type of a finite CW-complex. Let  $\text{aut}(X)$  denote the topological monoid of homotopy automorphisms of  $X$ , with the compact-open topology, and let  $\text{aut}_*(X)$  denote the submonoid of base-point preserving homotopy automorphisms. It is well known that the classifying space  $B \text{aut}(X)$  classifies fibrations with fiber  $X$ . Let us recall the precise meaning of this statement.

An  $X$ -fibration over a space  $B$  is a fibration  $E \rightarrow B$  such that for every point  $b \in B$  there is a homotopy equivalence  $X \rightarrow E_b$ . An elementary equivalence between two  $X$ -fibrations  $E \rightarrow B$  and  $E' \rightarrow B$  is a map  $E \rightarrow E'$  over  $B$  such that for every  $b \in B$  the induced map  $E_b \rightarrow E'_b$  is a homotopy equivalence. We let  $\mathcal{Fib}(B, X)$  denote the set of equivalence classes of  $X$ -fibrations over  $B$  under the equivalence relation generated by elementary equivalences.

**Theorem 3.1** (See [38]). *There is an  $X$ -fibration,*

$$(6) \quad E_X \rightarrow B_X.$$

*which is universal in the sense that the map*

$$[B, B_X] \rightarrow \mathcal{Fib}(B, X), \quad [\varphi] \mapsto [\varphi^*(E_X) \rightarrow B]$$

*is a bijection for every space  $B$  of the homotopy type of a CW-complex. Furthermore, the universal fibration (6) is weakly equivalent to the map*

$$B \text{aut}_*(X) \rightarrow B \text{aut}(X)$$

*induced by the inclusion of monoids  $\text{aut}_*(X) \rightarrow \text{aut}(X)$ .*

**3.2. Fibrations of dg Lie algebras.** There is a parallel story for dg Lie algebras. The category of dg Lie algebras has a model structure where the weak equivalences are the quasi-isomorphisms and the fibrations are the surjections. The cofibrant objects, i.e., the analogs of CW-complexes, are the dg Lie algebras whose underlying graded Lie algebra is free. Schlessinger and Stasheff gave an explicit construction of a classifying space for fibrations in this context, which we now will recall.

Let  $L$  be a dg Lie algebra. A derivation of degree  $p$  is a linear map  $\theta: L_* \rightarrow L_{*+p}$  such that

$$\theta[x, y] = [\theta(x), y] + (-1)^{p|x|}[x, \theta(y)],$$

for all  $x, y \in L$ . The derivations of  $L$  assemble into a dg Lie algebra  $\text{Der } L$ , whose Lie bracket and differential  $D$  are defined by

$$[\theta, \eta] = \theta \circ \eta - (-1)^{|\theta||\eta|}\eta \circ \theta, \quad D(\theta) = d \circ \theta - (-1)^{|\theta|}\theta \circ d,$$

where  $d$  is the differential in  $L$ .

Given a morphism of dg Lie algebras  $f: L \rightarrow L'$ , an  $f$ -derivation of degree  $p$  is a map  $\theta: L_* \rightarrow L'_{*+p}$  such that

$$\theta[x, y] = [\theta(x), f(y)] + (-1)^{p|x|}[f(x), \theta(y)],$$

for all  $x, y \in L$ . The  $f$ -derivations assemble into a chain complex  $\text{Der}_f(L, L')$ , whose differential  $D$  is defined by

$$D(\theta) = d_{L'} \circ \theta - (-1)^{|\theta|}\theta \circ d_L.$$

In general there is no natural Lie algebra structure on  $\text{Der}_f(L, L')$ .

The Jacobi identity for  $L$  implies that the map  $\text{ad}_x: L \rightarrow L$ , sending  $y$  to  $[x, y]$ , is a derivation of degree  $|x|$  for each  $x \in L$ . The map  $\text{ad}: L \rightarrow \text{Der } L$  sending  $x$  to  $\text{ad}_x$  is a morphism of dg Lie algebras. Let  $\text{Der } L // \text{ad } L$  denote the mapping cone of  $\text{ad}: L \rightarrow \text{Der } L$ , i.e.,

$$\text{Der } L // \text{ad } L = sL \oplus \text{Der } L,$$

with differential given by

$$\tilde{D}(\theta) = D(\theta), \quad \tilde{D}(sx) = \text{ad}_x - sd(x),$$

for  $\theta \in \text{Der } L$  and  $x \in L$ . There is a Lie bracket on  $\text{Der } L // \text{ad } L$ , which is defined as the extension of the Lie bracket on  $\text{Der } L$  that satisfies

$$[\theta, sx] = (-1)^{|\theta|}s\theta(x), \quad [sx, sy] = 0,$$

for  $\theta \in \text{Der } L$  and  $x, y \in L$ . The Schlessinger-Stasheff classifying dg Lie algebra of  $L$  is defined to be the positive truncation,

$$B_L = (\text{Der } L // \text{ad } L)^+.$$

Here, the positive truncation of a dg Lie algebra  $L$  is the sub dg Lie algebra  $L^+$  with

$$L_i^+ = \begin{cases} L_i, & i \geq 2 \\ \ker(d: L_1 \rightarrow L_0), & i = 1 \\ 0, & i \leq 0. \end{cases}$$

An  $L$ -fibration over  $K$  is a surjective map of dg Lie algebras  $\pi: E \rightarrow K$  together with a quasi-isomorphism  $L \rightarrow \text{Ker } \pi$ . An elementary equivalence between two  $L$ -fibrations  $\pi: E \rightarrow K$  and  $\pi': E' \rightarrow K$  is a quasi-isomorphism of dg Lie algebras  $E \rightarrow E'$  over  $K$  such that the diagram

$$\begin{array}{ccc} L & \xrightarrow{\quad} & \text{Ker } \pi \\ & \searrow & \downarrow \\ & & \text{Ker } \pi' \end{array}$$

commutes. Let  $\mathcal{Fib}(K, L)$  denote the set of equivalence classes of  $L$ -fibrations over  $K$  under the equivalence relation generated by elementary equivalence.



**Theorem 3.2** (See Tanré [50]). *Let  $L$  be a cofibrant dg Lie algebra and let  $B_L$  denote its Schlessinger-Stasheff classifying dg Lie algebra. There is an  $L$ -fibration*

$$(7) \quad E_L \rightarrow B_L,$$

*which is universal in the sense that for every cofibrant dg Lie algebra  $K$ , the map*

$$[K, B_L] \rightarrow \mathcal{Fib}(K, L), \quad [\varphi] \mapsto [\varphi^*(E_L)]$$

*is a bijection. Furthermore, the morphism  $E_L \rightarrow B_L$  is weakly equivalent to the morphism*

$$\mathrm{Der}^+ L \rightarrow (\mathrm{Der} L // \mathrm{ad} L)^+.$$

By combining Theorem 3.1 and Theorem 3.2, together with Quillen's equivalence of homotopy theories between  $\mathbf{Top}_{*,1}^{\mathbb{Q}}$  and  $\mathbf{DGL}$ , it is not difficult to derive the following consequence.

**Corollary 3.3** (See [50, Corollaire VII.4(4)]). *Let  $X$  be a simply connected space of the homotopy type of a finite CW-complex. Let  $\mathbb{L}_X$  be a cofibrant model of Quillen's dg Lie algebra  $\lambda(X)$ . The positive truncation of the morphism of dg Lie algebras*

$$\mathrm{Der} \mathbb{L}_X \rightarrow \mathrm{Der} \mathbb{L}_X // \mathrm{ad} \mathbb{L}_X$$

*is a dg Lie algebra model for the map of 1-connected covers*

$$B \mathrm{aut}_*(X)\langle 1 \rangle \rightarrow B \mathrm{aut}(X)\langle 1 \rangle.$$

**3.3. Relative fibrations.** Given a non-empty subspace  $A \subset X$ , we may consider the monoid  $\mathrm{aut}(X; A)$  of homotopy self-equivalences of  $X$  that restrict to the identity map on  $A$ . We will assume that the inclusion map from  $A$  into  $X$  is a cofibration. As follows from the theory of [38] (see, e.g., [27, Appendix B] for details), the classifying space  $B \mathrm{aut}(X; A)$  classifies fibrations with fiber  $X$  under the trivial fibration with fiber  $A$ .

Similarly, for a cofibration of cofibrant dg Lie algebras  $K \subset L$ , the positive truncation of the dg Lie algebra  $\mathrm{Der}(L; K)$  of derivations on  $L$  that restrict to zero on  $K$ , acts as a classifying space for fibrations of dg Lie algebras with fiber  $L$  under the trivial fibration with fiber  $K$ . This result seems not to have appeared in the literature, but the proof is a straightforward generalization of [50, Chapitre VII]. Details will appear elsewhere. The following is a consequence.

**Theorem 3.4.** *Let  $A \subset X$  be a cofibration of simply connected spaces of the homotopy type of finite CW-complexes, and let  $\mathbb{L}_A \subset \mathbb{L}_X$  be a cofibration between cofibrant dg Lie algebras that models the inclusion of  $A$  into  $X$ . Then the positive truncation of the dg Lie algebra  $\mathrm{Der}(\mathbb{L}_X; \mathbb{L}_A)$ , consisting of all derivations on  $\mathbb{L}_X$  that restrict to zero on  $\mathbb{L}_A$ , is a dg Lie algebra model for the simply connected cover of  $B \mathrm{aut}(X; A)$ .*

#### 4. AUTOMORPHISMS OF HIGHLY CONNECTED MANIFOLDS

In this section we will prove Theorem 1.6.

**4.1. Wall's classification of highly connected manifolds.** Let  $M$  be a closed oriented  $(d-1)$ -connected smooth manifold of dimension  $2d$ , where  $d \geq 3$ . The intersection form

$$\mu: H_d(M) \otimes H_d(M) \rightarrow \mathbb{Z}, \quad \mu(x, y) = \langle x, y \rangle,$$

endows  $H_d(M)$  with the structure of an  $(-1)^d$ -symmetric inner product space over  $\mathbb{Z}$ . Every homology class  $x \in H_d(M)$  may be represented as the fundamental class of some embedded sphere  $S^d \subset M$ . The normal bundle of the embedding

$S^d \subset M$  is classified by a map  $\nu: S^d \rightarrow BSO(d)$  and determines a homotopy class  $[\nu] \in \pi_{d-1}(SO(d))$ . The function

$$q: H_d(M) \rightarrow \pi_{d-1}(SO(d)), \quad x \mapsto [\nu],$$

is well-defined and satisfies the following equations:

$$(8) \quad q(x + y) = q(x) + q(y) + \langle x, y \rangle \partial(\iota_d),$$

$$(9) \quad HJq(x) = \langle x, x \rangle.$$

Here  $\partial(\iota_d) \in \pi_{d-1}(SO(d))$  denotes the image of the class of the identity map of  $S^d$  under the boundary map of the long exact homotopy sequence associated to the fibration  $SO(d) \rightarrow SO(d+1) \rightarrow S^d$ . In the second row,  $J: \pi_{d-1}(SO(d)) \rightarrow \pi_{2d-1}(S^d)$  is the  $J$ -homomorphism and  $H: \pi_{2d-1}(S^d) \rightarrow \mathbb{Z}$  the Hopf invariant. We refer to Wall's work [51, 52, 53, 54] for more details.

By a *(geometric) quadratic module* we will mean the data  $(H, \mu, q)$  of an abelian group  $H$  together with a  $(-1)^d$ -symmetric non-degenerate bilinear form

$$\mu: H \otimes H \rightarrow \mathbb{Z}, \quad \mu(x, y) = \langle x, y \rangle,$$

and a function

$$q: H \rightarrow \pi_{d-1}(SO(d)),$$

such that the equations (8) and (9) are satisfied. A morphism of quadratic modules is a homomorphism that preserves  $\mu$  and  $q$ . Let  $Q(M)$  denote the quadratic module  $(H_d(M), \mu, q)$  associated to a highly connected manifold  $M$ .

If the normal bundles  $\nu$  of the embedded spheres  $S^d \subset M$  are stably trivial, i.e., if the tangent bundle  $\tau_M$  restricts to the trivial bundle on the embedded spheres, then the quadratic function  $q$  maps into the subgroup of  $\pi_{d-1}(SO(d))$  generated by  $\partial(\iota_d)$ . The  $J$ -homomorphism maps this subgroup isomorphically onto the subgroup of  $\pi_{2d-1}(S^d)$  generated by the Whitehead product  $[\iota_d, \iota_d]$ . If  $d$  is even, then  $\partial(\iota_d)$  has infinite order, and in this case the quadratic function is determined by the self-intersection by (9). If  $d$  is odd and  $\neq 1, 3, 7$ , then  $\partial(\iota_d)$  is a non-zero element of order 2. In the cases  $d = 1, 3, 7$ , we have  $\partial(\iota_d) = 0$ .

Let  $N$  denote the manifold obtained by removing an open  $2d$ -disk from  $M$ . Then  $N$  is homotopy equivalent to a wedge of spheres  $\vee^n S^d$ , where  $n$  is the rank of  $H = H_d(M)$ , and we may identify its boundary  $\partial N$  with  $S^{2d-1}$ . The homotopy type of  $M$  is determined by the homotopy class  $\omega \in \pi_{2d-1}(N)$  of the inclusion  $S^{2d-1} = \partial N \rightarrow N$ , which may be expressed in terms of the associated quadratic module as follows. Let  $\alpha_i: S^d \rightarrow N$ , for  $i = 1, \dots, n$ , represent a basis for  $\pi_d(N)$  and let  $e_1, \dots, e_n$  be the corresponding basis for  $H$ . Then we have the equality

$$(10) \quad \omega = \sum_{i < j} \langle e_i, e_j \rangle [\alpha_i, \alpha_j] + \sum_i \alpha_i \circ Jq(e_i)$$

of elements in the homotopy group  $\pi_{2d-1}(N)$ , see [51]. We note furthermore that the rational homotopy groups  $\pi_*^{\mathbb{Q}}(N) = \pi_{*+1}(N) \otimes \mathbb{Q}$ , with the Whitehead product, is a free graded Lie algebra on the classes  $\alpha_1, \dots, \alpha_n$ .

**4.2. Mapping class groups.** The mapping class groups of highly connected manifolds may be described in terms of the associated quadratic modules, up to extensions. We will recall the calculation for the homotopy and block diffeomorphism mapping class groups of  $N$  relative to its boundary, see [9] and [30, Proposition 3].

**Proposition 4.1.** *Let  $d \geq 3$ . There is a commutative diagram with exact rows*

$$\begin{array}{ccccccc} 0 & \longrightarrow & \tilde{K} & \longrightarrow & \pi_0 \widetilde{\text{Diff}}_{\partial}(N) & \longrightarrow & \text{Aut}(H, \mu, q) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & K & \longrightarrow & \pi_0 \text{aut}_{\partial}(N) & \longrightarrow & \text{Aut}(H, \mu, Jq) \longrightarrow 0. \end{array}$$

The group  $K$  is finite. The group  $\tilde{K}$  is finite except when  $d \equiv 3 \pmod{4}$ , in which case there is an exact sequence

$$0 \longrightarrow \theta_{2d+1} \longrightarrow \tilde{K} \longrightarrow H \longrightarrow 0,$$

where  $\theta_{2d+1}$  denotes the group of  $(2d+1)$ -dimensional homotopy spheres.

**Example 4.2.** For the manifold  $S^d \times S^d$  the normal bundles of the embeddings  $S^d \times * \subset S^d \times S^d$  and  $* \times S^d \subset S^d \times S^d$  are trivial. Thus, if we let  $e$  and  $f$  be the classes in  $H_d(S^d \times S^d)$  represented by these embeddings, then the quadratic module associated to  $S^d \times S^d$  is given by  $(\mathbb{Z}e \oplus \mathbb{Z}f, \mu, q)$ , where

$$\langle e, e \rangle = 0, \quad \langle e, f \rangle = 1, \quad \langle f, f \rangle = 0,$$

$$q(ae + bf) = ab\partial(\iota_d).$$

Connected sums of oriented manifolds correspond to orthogonal sums of quadratic modules; for  $M$  and  $N$  two highly connected manifolds, there is a natural isomorphism of quadratic modules  $Q(M \# N) \cong Q(M) \oplus Q(N)$ . It follows that the quadratic module associated to the manifold  $M_g = \#^g S^d \times S^d$  is the hyperbolic module  $(H_g, \mu, q)$ : there is a basis  $e_1, \dots, e_g, f_1, \dots, f_g$  for  $H_g$  such that

$$\langle e_i, e_j \rangle = 0, \quad \langle e_i, f_j \rangle = \delta_{ij}, \quad \langle f_i, f_j \rangle = 0,$$

$$q(a_1 e_1 + \dots + a_g e_g + b_1 f_1 + \dots + b_g f_g) = \sum_{i=1}^g a_i b_i \partial(\iota_d).$$

It follows that  $\text{Aut}(H_g, \mu, q) = \text{Aut}(H_g, \mu, Jq)$  for the hyperbolic module. The automorphism group

$$\Gamma_g := \text{Aut}(H_g, \mu, q)$$

admits the following concrete description. If  $d$  is even, then  $\Gamma_g$  is isomorphic to the automorphism group  $\text{Aut}(H_g, \mu)$ , i.e., to the orthogonal group  $O_{g,g}(\mathbb{Z})$ . If  $d = 1, 3, 7$ , then  $\Gamma_g$  is isomorphic to the symplectic group  $\text{Sp}_{2g}(\mathbb{Z})$ . If  $d \neq 1, 3, 7$  is odd, then  $\Gamma_g$  is isomorphic to the subgroup of  $\text{Sp}_{2g}(\mathbb{Z})$  consisting of those symplectic matrices

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

for which the diagonal entries of the  $g \times g$ -matrices  $\gamma^t \alpha$  and  $\delta^t \beta$  are even. For this last description, see e.g., [5, §3]. In the notation of [5],  $\Gamma_g$  is isomorphic to the automorphism group of the hyperbolic module in the category  $Q^\lambda(A, \Lambda)$ , where  $A$  is the ring  $\mathbb{Z}$  with trivial involution,  $\lambda = (-1)^d$ , and  $\Lambda = 0$  if  $d$  is even,  $\Lambda = \mathbb{Z}$  if  $d = 1, 3, 7$  and  $\Lambda = 2\mathbb{Z}$  if  $d \neq 1, 3, 7$  is odd.

In what follows we will describe the rational homotopy of the simply connected cover of  $B \text{aut}_{\partial}(N)$ , viewed as a representation of the mapping class group.

**4.3. Equivariant rational homotopy type.** Let  $M$  be a  $(d-1)$ -connected,  $2d$ -dimensional manifold and let  $N$  be the manifold obtained by removing an open  $2d$ -disk from  $M$ . Let  $(H, \mu, q)$  be the associated quadratic module and let  $H^{\mathbb{Q}} = H \otimes \mathbb{Q}$ . We may identify  $\pi_d(N)$  with  $H$ , and the homotopy Lie algebra  $\pi_*^{\mathbb{Q}}(N) = \pi_{*+1}(N) \otimes \mathbb{Q}$  with the free graded Lie algebra  $\mathbb{L}(H^{\mathbb{Q}})$ , where the generators are put in degree  $d-1$ . We let

$$\mathrm{Der}_{\omega}^{+} \mathbb{L}(H^{\mathbb{Q}})$$

denote the graded Lie algebra of positive degree derivations on  $\mathbb{L}(H^{\mathbb{Q}})$  that annihilate the element  $\omega$ .

**Theorem 4.3.** *Consider the classifying space  $X = B\mathrm{aut}_{\partial}(N)$ . If  $d \geq 3$  and  $\mathrm{rank} H > 2$ , then the following holds.*

- (1) *The group  $\pi_1(X)$  surjects onto  $\mathrm{Aut}(H, \mu, Jq)$  with finite kernel.*
- (2) *Quillen's dg Lie algebra  $\lambda(\tilde{X})$  is formal.*
- (3) *There is a  $\pi_1(X)$ -equivariant isomorphism of graded Lie algebras*

$$\pi_*^{\mathbb{Q}}(\tilde{X}) \cong \mathrm{Der}_{\omega}^{+} \mathbb{L}(H^{\mathbb{Q}}),$$

*where the action on the right hand side is induced by the natural action of  $\mathrm{Aut}(H, \mu, Jq)$  on  $H$ .*

*Proof.* The first statement follows from Proposition 4.1 since  $\pi_1(X) \cong \pi_0 \mathrm{aut}_{\partial}(N)$ .

Cofibrant models for the dg Lie algebras  $\lambda(\partial N)$  and  $\lambda(N)$  are given by the free graded Lie algebras, with trivial differentials,  $\mathbb{L}(\rho)$  on a generator of degree  $2d-2$ , and  $\mathbb{L}(H^{\mathbb{Q}}) = \mathbb{L}(\alpha_1, \dots, \alpha_n)$  on  $n = \mathrm{rank} H$  generators of degree  $d-1$ , respectively. A model for the inclusion map  $\partial N \rightarrow N$  is given by the morphism of graded Lie algebras

$$\varphi: \mathbb{L}(\rho) \rightarrow \mathbb{L}(\alpha_1, \dots, \alpha_n), \quad \rho \mapsto \omega,$$

where, by (10),

$$\omega = \frac{1}{2} \sum_{i,j} \langle e_i, e_j \rangle [\alpha_i, \alpha_j].$$

(To see this, note that  $Jq(e_i) = \frac{1}{2} \langle e_i, e_i \rangle [\iota_d, \iota_d]$  rationally.)

We would like to use Theorem 3.4, but  $\varphi$  is not a cofibration of dg Lie algebras. To rectify this, we factor  $\varphi$  as a cofibration  $i$  followed by a surjective quasi-isomorphism  $p$  as follows:

$$\mathbb{L}(\rho) \xrightarrow{i} (\mathbb{L}(\rho, \alpha_1, \dots, \alpha_n, \gamma), \delta(\gamma) = \omega - \rho) \xrightarrow{p} \mathbb{L}(\alpha_1, \dots, \alpha_n).$$

Here  $i$  is the obvious inclusion, and  $p$  is defined by  $p(\rho) = \omega$ ,  $p(\alpha_i) = \alpha_i$  and  $p(\gamma) = 0$ . Now,  $i$  is a cofibration that models the inclusion of  $\partial N$  into  $N$ . By Theorem 3.4, the dg Lie algebra

$$(11) \quad \mathrm{Der}^{+}(\mathbb{L}(\rho, \alpha_1, \dots, \alpha_n, \gamma); \mathbb{L}(\rho))$$

is a model for the simply connected cover of  $B\mathrm{aut}_{\partial}(N)$ . Thus, to prove (2) we need to show that the dg Lie algebra (11) is formal. To simplify notation, let  $\mathbb{L}_N = \mathbb{L}(\alpha_1, \dots, \alpha_n)$ ,  $\mathbb{L}_{\partial N} = \mathbb{L}(\rho)$  and  $\mathbb{L} = \mathbb{L}(\rho, \alpha_1, \dots, \alpha_n, \gamma)$ .

By the same argument as in Lemma 4.6 and Lemma 4.7 below, the surjective quasi-isomorphism  $p: \mathbb{L} \rightarrow \mathbb{L}_N$  gives rise to an  $L_{\infty}$ -morphism  $\mathrm{Der} \mathbb{L}_N \rightarrow \mathrm{Der} \mathbb{L}$ , which restricts to an  $L_{\infty}$ -morphism

$$\psi: \mathrm{Der}^{+}(\mathbb{L}_N; \mathbb{L}_{\partial N}) \rightarrow \mathrm{Der}^{+}(\mathbb{L}; \mathbb{L}_{\partial N})$$

such that the diagram of chain maps

$$\begin{array}{ccc} \mathrm{Der}^+(\mathbb{L}; \mathbb{L}_{\partial N}) & \xrightarrow{p_*} & \mathrm{Der}_p^+(\mathbb{L}, \mathbb{L}_N; \mathbb{L}_{\partial N}) \\ & \searrow \psi & \uparrow p^* \\ & & \mathrm{Der}^+(\mathbb{L}_N; \mathbb{L}_{\partial N}) \end{array}$$

commutes, and such that  $p_*$  is a quasi-isomorphism. Thus,  $\psi$  is a quasi-isomorphism if and only if  $p^*$  is. That  $p^*$  is a quasi-isomorphism can be seen by considering the following diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{Der}^+(\mathbb{L}_N; \mathbb{L}_{\partial N}) & \longrightarrow & \mathrm{Der}^+ \mathbb{L}_N & \xrightarrow{\varphi^*} & \mathrm{Der}_{\varphi}^+(\mathbb{L}_{\partial N}, \mathbb{L}_N) \longrightarrow 0 \\ & & \downarrow p^* & & \downarrow p^* & & \parallel \\ 0 & \longrightarrow & \mathrm{Der}_p^+(\mathbb{L}, \mathbb{L}_N; \mathbb{L}_{\partial N}) & \longrightarrow & \mathrm{Der}_p^+(\mathbb{L}, \mathbb{L}_N) & \xrightarrow{i^*} & \mathrm{Der}_{\varphi}^+(\mathbb{L}_{\partial N}, \mathbb{L}_N) \longrightarrow 0. \end{array}$$

The map  $i^*$  in the bottom row is surjective because  $i$  is a cofibration. Surjectivity of the map  $\varphi^*$  in the top row is more subtle, and it depends crucially on non-degeneracy of the intersection form. In fact, we will see below in Proposition 5.4 that non-degeneracy of the intersection form implies that the map  $\varphi^*$  is isomorphic to the bracketing map on the free Lie algebra,

$$H^{\mathbb{Q}} \otimes \mathbb{L}^{\geq 2}(H^{\mathbb{Q}}) \rightarrow \mathbb{L}^{\geq 3}(H^{\mathbb{Q}}),$$

which is evidently surjective. This middle vertical map  $p^*$  is a quasi-isomorphism because  $p$  is a quasi-isomorphism between cofibrant dg Lie algebras. By the five lemma the left vertical map is a quasi-isomorphism. This finishes the proof of (2).

In particular, there is an isomorphism of graded Lie algebras

$$(12) \quad \pi_*^{\mathbb{Q}}(B \mathrm{aut}_{\partial}(N)) \cong \mathrm{Der}_{\omega}^+ \mathbb{L}(H^{\mathbb{Q}}).$$

To calculate the action of the fundamental group  $\pi_1(B \mathrm{aut}_{\partial}(N))$  on  $\pi_*^{\mathbb{Q}}(B \mathrm{aut}_{\partial}(N))$ , we need to make the isomorphism (12) explicit.

Let  $G$  be a topological group with neutral element  $e$  as base-point. The Samelson product

$$\pi_p(G) \times \pi_q(G) \rightarrow \pi_{p+q}(G)$$

is a natural operation on the homotopy groups of  $G$  defined as follows. Given based maps  $f: S^p \rightarrow G$  and  $g: S^q \rightarrow G$ , the composite map

$$S^p \times S^q \xrightarrow{(f,g)} G \times G \xrightarrow{[-,-]} G$$

is trivial when restricted to  $S^p \vee S^q$ , so defines a based map  $[f, g]: S^{p+q} \cong S^p \times S^q / S^p \vee S^q \rightarrow G$ , the homotopy class of which represents the Samelson product of the classes  $[f]$  and  $[g]$ . The map  $G \rightarrow G$  sending  $x$  to  $gxg^{-1}$  preserves the base-point, and defines a homomorphism  $\phi_g: \pi_k(G) \rightarrow \pi_k(G)$ . This defines an action of the group  $\pi_0(G)$  on  $\pi_k(G)$ , and this action preserves Samelson products. Under the standard isomorphism  $\pi_{k+1}(BG) \cong \pi_k(G)$ , the Whitehead product on  $\pi_{*+1}(BG)$  corresponds to the Samelson product on  $\pi_*(G)$ , and the usual action of  $\pi_1(BG)$  on  $\pi_{k+1}(BG)$  corresponds to action of  $\pi_0(G)$  on  $\pi_k(G)$  described above, see [55]. The above holds true for  $G$  a group-like topological monoid, because every such may be replaced by a homotopy equivalent group. In particular it applies to monoids of homotopy automorphisms.

Let  $f: X \rightarrow Y$  be a map between simply connected based spaces, and let  $\varphi: \mathbb{L}_X \rightarrow \mathbb{L}_Y$  be a Lie model for  $f$ . There is a natural bijection for all  $k \geq 1$ ,

$$(13) \quad \pi_k(\mathrm{map}_*(X, Y), f) \otimes \mathbb{Q} \cong H_k(\mathrm{Der}_{\varphi}(\mathbb{L}_X, \mathbb{L}_Y)),$$

which is an isomorphism of  $\mathbb{Q}$ -vector spaces for  $k \geq 2$ , see [33]. If  $f = id_X$ , it is a vector space isomorphism also for  $k = 1$ , and under the isomorphism

$$(14) \quad \pi_k(\text{aut}_*(X), id_X) \otimes \mathbb{Q} \cong H_k(\text{Der}(\mathbb{L}_X)),$$

the Samelson product corresponds to the Lie bracket on derivations.

The isomorphism (13) may be described as follows. A map  $h: S^k \rightarrow \text{map}_*(X, Y)$  sending the base-point of  $S^k$  to  $f$  is the same as a map  $g$  making the diagram

$$\begin{array}{ccc} X & \xrightarrow{g} & \text{map}(S^k, Y) \\ & \searrow f & \swarrow ev_* \\ & Y & \end{array}$$

commute. The mapping space  $\text{map}(S^k, Y)$  has Lie model  $H^*(S^k) \otimes \mathbb{L}_Y$ . If we let  $\chi: \mathbb{L}_X \rightarrow H^*(S^k) \otimes \mathbb{L}_Y$  be a Lie model for  $g$ , then we can define a map  $\theta: \mathbb{L}_X \rightarrow \mathbb{L}_Y$  of degree  $k$  by

$$\chi(\xi) = 1 \otimes \varphi(\xi) + [S^k] \otimes \theta(\xi).$$

The assertion that  $\chi$  is a morphism of dg Lie algebras is equivalent to  $\theta$  being a  $\varphi$ -derivation and a cycle. The isomorphism (13) is effected by the map

$$\pi_k(\text{map}_*(X, Y), f) \rightarrow H_k(\text{Der}_\varphi(\mathbb{L}_X, \mathbb{L}_Y)), \quad [h] \mapsto [\theta].$$

A Lie model for  $N$  is the free graded Lie algebra  $\mathbb{L}_N = \mathbb{L}(H^\mathbb{Q})$  with trivial differential. The generators have degree  $d - 1$ . According to (14), there is an isomorphism of graded Lie algebras

$$(15) \quad \pi_*(\text{aut}_*(N), id_N) \otimes \mathbb{Q} \cong \text{Der}^+ \mathbb{L}(H^\mathbb{Q}).$$

The action of  $\pi_0(\text{aut}_*(N))$  on  $\pi_*(\text{aut}_*(N)) \otimes \mathbb{Q}$  may be identified by exploiting the naturality of the isomorphism (13). Indeed, if  $f: N \rightarrow N$  is a based self-equivalence, then a Lie model for  $f$  is simply given by the map  $\varphi_f: \mathbb{L}_N \rightarrow \mathbb{L}_N$  that is induced by  $f$  in homotopy. By our previous considerations, the action of the class  $[f] \in \pi_0(\text{aut}_*(N))$  on  $\pi_*(\text{aut}_*(N)) \otimes \mathbb{Q}$  is induced by the self-map of  $\text{aut}_*(N)$  that sends  $g$  to  $f g f^{-1}$ , where  $f^{-1}$  is a choice of homotopy inverse of  $f$ . From the naturality properties of the isomorphism (13) it follows that the action of  $[f]$  on  $\text{Der}^+(\mathbb{L}_N)$  is given by

$$(16) \quad \theta \mapsto \varphi_f \circ \theta \circ \varphi_f^{-1}.$$

The monoid  $\text{aut}_\partial(N)$  is the fiber over the inclusion map  $\partial N \rightarrow N$  under the fibration

$$i^*: \text{aut}_*(N) \rightarrow \text{map}_*(\partial N, N),$$

induced by the inclusion  $i: \partial N \rightarrow N$ . By (13), the map in rational homotopy induced by  $i^*$  may be identified with the map

$$(17) \quad \varphi^*: \text{Der}^+ \mathbb{L}_N \rightarrow \text{Der}_\varphi^+(\mathbb{L}(\rho), \mathbb{L}(H^\mathbb{Q}))$$

induced by the morphism  $\varphi: \mathbb{L}(\rho) \rightarrow \mathbb{L}(H^\mathbb{Q})$ ,  $\rho \mapsto \omega$ , which appeared before. As we already pointed out, the map (17) is surjective. It follows that the rational homotopy exact sequence of the fibration

$$\text{aut}_\partial(N) \rightarrow \text{aut}_*(N) \rightarrow \text{map}_*(\partial N, N)$$

splits into short exact sequences, and the rational homotopy groups of  $\text{aut}_\partial(N)$  may be identified with the kernel of (17). Thus, again we see that, for  $* > 0$ ,

$$(18) \quad \pi_*(\text{aut}_\partial(N)) \otimes \mathbb{Q} \cong \text{Der}_\omega^+ \mathbb{L}(H^\mathbb{Q}).$$

However, this isomorphism is a priori not the same as the one obtained before, and an argument is needed to show that it commutes with Lie brackets. But

since  $\text{aut}_\partial(N) \rightarrow \text{aut}_*(N)$  is a map of monoids the map  $\pi_*(\text{aut}_\partial(N)) \otimes \mathbb{Q} \rightarrow \pi_*(\text{aut}_*(N)) \otimes \mathbb{Q}$  commutes with Samelson products, and since the map is injective, we may calculate Lie brackets in  $\pi_*(\text{aut}_\partial(N)) \otimes \mathbb{Q}$  by passing to  $\pi_*(\text{aut}_*(N)) \otimes \mathbb{Q}$ , where they are calculated in terms of derivations (15), so it does follow that (18) preserves Lie brackets. By a similar argument, the action of  $\pi_0(\text{aut}_\partial(N))$  is described by (16).  $\square$

**Corollary 4.4.** *The rational cohomology groups of  $B\text{aut}_\partial(N)$  are finite dimensional in each degree.*

*Proof.* Let  $X = B\text{aut}_\partial(N)$ . The graded Lie algebra  $\text{Der}_\omega^+ \mathbb{L}(H^\mathbb{Q})$  is finite dimensional in each degree, so by Theorem 4.3, the rational homotopy groups of  $\tilde{X}$  are finite dimensional in each degree. Hence, the same is true of the rational cohomology groups of  $\tilde{X}$ . The fundamental group  $\pi_1(X)$  is arithmetic, so the cohomology  $H^p(\pi_1(X); V)$  in any finite dimensional representation  $V$  is finite dimensional, see Theorem A.1. Thus, in the universal cover spectral sequence,

$$E_2^{p,q} = H^p(\pi_1(X); H^q(\tilde{X}; \mathbb{Q})) \Rightarrow H^{p+q}(X; \mathbb{Q}),$$

each term  $E_2^{p,q}$  is finite dimensional. It follows that  $H^k(X; \mathbb{Q})$  is finite dimensional for every  $k$ .  $\square$

**4.4. Free and based homotopy automorphisms.** We now turn to the rational homotopy theory of the classifying spaces of the monoids  $\text{aut}(M)$  and  $\text{aut}_*(M)$  of free and base-point preserving homotopy automorphisms, respectively, for highly connected manifolds  $M$ .

Let  $M$  be a closed  $(d-1)$ -connected,  $2d$ -dimensional manifold, and let  $N$  be the result of removing an open  $2d$ -disk from  $M$ . Recall from §4.1 the definition of the quadratic module  $(H, \mu, q)$  and the homotopy class  $\omega \in \pi_{2d-1}(N)$ . Let  $n$  be the rank of  $H$ . The rational homotopy groups  $\pi_*^\mathbb{Q}(N) = \pi_{*+1}(N) \otimes \mathbb{Q}$ , with the Whitehead product, may be identified with the free graded Lie algebra  $\mathbb{L}(\alpha_1, \dots, \alpha_n)$  over  $\mathbb{Q}$  on classes  $\alpha_1, \dots, \alpha_n$  of degree  $d-1$ . Under this identification, the homotopy class  $\omega$  assumes the form

$$\omega = \frac{1}{2} \sum_{i,j} \langle e_i, e_j \rangle [\alpha_i, \alpha_j].$$

The rational homotopy groups of the closed manifold  $M$  may be identified with the quotient graded Lie algebra,

$$\pi_*^\mathbb{Q}(M) \cong \mathbb{L}(\alpha_1, \dots, \alpha_n) / (\omega).$$

**Theorem 4.5.** *Let  $M$  be a closed  $(d-1)$ -connected  $2d$ -dimensional manifold, where  $d \geq 3$ , and consider the classifying spaces*

$$X = B\text{aut}(M), \quad X_* = B\text{aut}_*(M).$$

*If  $\text{rank } H > 2$ , then*

- (1) *Both groups  $\pi_1(X)$  and  $\pi_1(X_*)$  surject onto  $\text{Aut}(H, \mu, Jq)$  with finite kernel.*
- (2) *The Quillen dg Lie algebras  $\lambda(\tilde{X})$  and  $\lambda(\tilde{X}_*)$  are formal.*
- (3) *There are  $\pi_1$ -equivariant isomorphisms of graded Lie algebras*

$$\begin{aligned} \pi_*^\mathbb{Q}(\tilde{X}) &\cong \text{OutDer}^+(\mathbb{L}(\alpha_1, \dots, \alpha_n) / (\omega)), \\ \pi_*^\mathbb{Q}(\tilde{X}_*) &\cong \text{Der}^+(\mathbb{L}(\alpha_1, \dots, \alpha_n) / (\omega)). \end{aligned}$$

Statement (1) about the homotopy mapping class groups  $\pi_1(X) = \pi_0 \text{aut}(M)$  and  $\pi_1(X_*) = \pi_0 \text{aut}_*(M)$  was established in [9], see also [7]. The proof of (3) is similar to the proof of Theorem 4.3 (3).

Let  $L = \mathbb{L}(\alpha_1, \dots, \alpha_n)/(\omega)$ . A cofibrant dg Lie algebra model for  $\lambda(M)$  is given by

$$\mathbb{L} = (\mathbb{L}(\alpha_1, \dots, \alpha_n, \rho), \delta), \quad \delta(\rho) = \omega, \quad \delta(\alpha_i) = 0.$$

The generators  $\alpha_i$  have degree  $d-1$  and the generator  $\rho$  has degree  $2d-1$ . There is an evident surjective morphism of dg Lie algebras  $f: \mathbb{L} \rightarrow L$ , which is easily seen to be a quasi-isomorphism. Since we are working with positively graded chain complexes over a field, it is possible to extend  $f$  to a *contraction*, i.e., a diagram

$$(19) \quad \begin{array}{ccc} & f & \\ h \circlearrowleft & \mathbb{L} & \xrightarrow{\quad} L \\ & g & \end{array}$$

where  $g$  is a chain map with  $fg = 1_L$ , and  $h$  is a homotopy between  $gf$  and  $1_{\mathbb{L}}$ , i.e.,  $dh + hd = gf - 1_{\mathbb{L}}$ . We may without loss of generality assume that  $fh = 0$ ,  $hg = 0$  and  $h^2 = 0$ . The maps  $g$  and  $h$  will in general not be compatible with the Lie brackets.

Let  $\text{Der}_f(\mathbb{L}, L)$  denote the chain complex of  $f$ -derivations. Its elements of degree  $p$  are by definition all maps  $\theta: \mathbb{L} \rightarrow L$  of degree  $p$  that satisfy

$$\theta[x, y] = [\theta(x), f(y)] + (-1)^{|x|p}[f(x), \theta(y)]$$

for all  $x, y \in \mathbb{L}$ . The differential  $D$  is defined by

$$D(\theta) = d_L \circ \theta - (-1)^p \theta \circ d_{\mathbb{L}}.$$

There is an obvious chain map  $f_*: \text{Der } \mathbb{L} \rightarrow \text{Der}_f(\mathbb{L}, L)$ .

**Lemma 4.6.** *Given a contraction as in (19), where  $f$  is a morphism of dg Lie algebras and the underlying graded Lie algebra of  $\mathbb{L}$  is free, there is an induced contraction*

$$h' \circlearrowleft \begin{array}{ccc} & f_* & \\ \text{Der}(\mathbb{L}) // \text{ad } \mathbb{L} & \xrightarrow{\quad} & \text{Der}_f(\mathbb{L}, L) // \text{ad } L \\ & g' & \end{array}$$

In particular,  $f_*$  is a quasi-isomorphism.

*Proof.* Say  $\mathbb{L} = (\mathbb{L}(V), d)$ . Since a derivation on a free Lie algebra is determined by its restriction to generators, there are isomorphisms of graded vector spaces

$$\text{Der}(\mathbb{L}) // \text{ad } \mathbb{L} \cong s\mathbb{L} \oplus \text{Hom}(V, \mathbb{L}), \quad \text{Der}_f(\mathbb{L}, L) // \text{ad } L \cong sL \oplus \text{Hom}(V, L).$$

We may write the differential of  $\text{Der}_f(\mathbb{L}, L) // \text{ad } L$  as  $\tilde{D} = d_* + t$ , where

$$d_*(sx) = -sd(x), \quad d_*(\theta) = d \circ \theta$$

and

$$t(sx) = \text{ad}_x, \quad t(\theta) = -(-1)^{|\theta|} \theta \circ d.$$

Similarly for  $\text{Der}(\mathbb{L}) // \text{ad } \mathbb{L}$ . The contraction (19) induces a contraction

$$h_* \circlearrowleft \begin{array}{ccc} & f_* & \\ s\mathbb{L} \oplus \text{Hom}(V, \mathbb{L}), d_* & \xrightarrow{\quad} & (sL \oplus \text{Hom}(V, L), d_*) \\ & g_* & \end{array}$$

By the basic perturbation lemma we obtain a new contraction

$$h' \circlearrowleft \begin{array}{ccc} & f' & \\ (s\mathbb{L} \oplus \text{Hom}(V, \mathbb{L}), d_* + t) & \xrightarrow{\quad} & (sL \oplus \text{Hom}(V, L), d_* + t') \\ & g' & \end{array}$$

where the new maps are defined by the recursive formulas

$$\begin{aligned} f' &= f_* + f' t h_*, & g' &= g_* + h_* t g', \\ h' &= h_* + h' t h_*, & t' &= f' t g_*. \end{aligned}$$



Since  $f$  is a morphism of Lie algebras, we have that  $f_*t = tf_*$ . Therefore  $f' = f_*$  and  $t' = f'tg_* = t$ , and the above may be identified with the sought after contraction between derivation complexes.  $\square$

We have a diagram of chain complexes

$$\begin{array}{ccc} h' \circlearrowleft \text{Der}(\mathbb{L}) // \text{ad } \mathbb{L} & \xrightleftharpoons[f_*]{g'} & \text{Der}_f(\mathbb{L}, L) // \text{ad } L \\ & \searrow \psi & \uparrow f^* \\ & & \text{Der}(L) // \text{ad } L \end{array}$$

**Lemma 4.7.** *Given a contraction as in (19) where  $f$  is a morphism of dg Lie algebras and the underlying graded Lie algebra of  $\mathbb{L}$  is free, the chain map*

$$\psi = g' \circ f^* : \text{Der}(L) // \text{ad } L \rightarrow \text{Der}(\mathbb{L}) // \text{ad } \mathbb{L}$$

*extends to an  $L_\infty$ -morphism.*

*Proof.* We will define the  $L_\infty$  morphism  $\{\psi_n\}_n$  by induction. We start with  $\psi_1 = \psi$ . Consider the chain map

$$\lambda_2 : (\text{Der}(L) // \text{ad } L)^{\otimes 2} \rightarrow (\text{Der}(\mathbb{L}) // \text{ad } \mathbb{L}), \quad \lambda_2(\theta, \eta) = \psi[\theta, \eta] - [\psi(\theta), \psi(\eta)].$$

Since  $f$  is a morphism of dg Lie algebras and  $f_* \circ \psi = f^*$ , we have that  $f_* \circ \lambda_2 = 0$ . Therefore, it follows from the relation

$$h'D + Dh' = g'f_* - 1$$

that  $\lambda_2 = \partial(-h' \circ \lambda_2)$  in the chain complex  $\text{Hom}((\text{Der}(L) // \text{ad } L)^{\otimes 2}, \text{Der}(\mathbb{L}) // \text{ad } \mathbb{L})$ . So we set  $\psi_2 := -h'\lambda_2$ . Since  $h'g' = 0$  we have that

$$\psi_2(\theta, \eta) = h'[\psi(\theta), \psi(\eta)].$$

We proceed by induction. In general,  $\psi_n : (\text{Der}(L) // \text{ad } L)^{\otimes n} \rightarrow \text{Der}(\mathbb{L}) // \text{ad } \mathbb{L}$  is an alternating sum over all binary trees with  $n$  leaves, where the leaves are decorated by  $\psi_1$ , the internal edges and the root are decorated by  $h'$  and the vertices are decorated by the Lie bracket in  $\text{Der}(\mathbb{L}) // \text{ad } \mathbb{L}$ .  $\square$

**Theorem 4.8.** *Let  $M$  be a simply connected space of finite  $\mathbb{Q}$ -type. Suppose that  $\lambda(M)$  is formal, i.e., that there is a quasi-isomorphism  $f : \mathbb{L} \rightarrow L$  from the minimal cofibrant model  $\mathbb{L}$  of  $\lambda(M)$  to the rational homotopy Lie algebra  $L = \pi_*^{\mathbb{Q}}(M)$ . If the center of  $L$  is trivial, and if the map  $f^* : \text{Der } L \rightarrow \text{Der}_f(\mathbb{L}, L)$  induces an isomorphism in homology in non-negative degrees, then the homotopy fiber sequence*

$$M \rightarrow B \text{aut}_*(M)\langle 1 \rangle \rightarrow B \text{aut}(M)\langle 1 \rangle$$

*is rationally modeled by the short exact sequence of graded Lie algebras*

$$0 \rightarrow L \xrightarrow{\text{ad}} \text{Der}^+ L \rightarrow \text{Der}^+ L / \text{ad } L \rightarrow 0.$$

*Proof.* A five lemma argument shows that if  $f^* : \text{Der } L \rightarrow \text{Der}_f(\mathbb{L}, L)$  induces an isomorphism in homology in non-negative degrees, then  $f^* : \text{Der}^+ L // \text{ad } L \rightarrow \text{Der}_f^+(\mathbb{L}, L) // \text{ad } L$  is a quasi-isomorphism. Since  $g'$  is a quasi-isomorphism, this implies that the  $L_\infty$ -morphisms  $\psi$  and  $\psi_1$  in Lemma 4.7 are quasi-isomorphisms after truncating. By Theorem 3.2, this shows that the map  $\text{Der}^+ L \rightarrow \text{Der}^+ L // \text{ad } L$  is a model for the map  $p : B \text{aut}_*(M)\langle 1 \rangle \rightarrow B \text{aut}(M)\langle 1 \rangle$ . If the center of  $L$  is trivial, then the morphism  $\text{Der } L // \text{ad } L \rightarrow \text{Der } L / \text{ad } L$  is a quasi-isomorphism, and hence the surjection  $\text{Der}^+ L \rightarrow \text{Der}^+ L / \text{ad } L$  is a model for  $p$ . By [23, Proposition 24.8], it follows that the short exact sequence

$$0 \rightarrow L \rightarrow \text{Der}^+ L \rightarrow \text{Der}^+ L / \text{ad } L \rightarrow 0$$

is a model for the fibration sequence  $M \rightarrow \text{Baut}_*(M)\langle 1 \rangle \rightarrow \text{Baut}(M)\langle 1 \rangle$ , as claimed.  $\square$

To finish the proof of Theorem 4.5, it suffices to verify the hypotheses of Theorem 4.8. This is done in Proposition 4.9 and Lemma 4.10 below.

**Proposition 4.9.** *Let  $M$  be a  $(d-1)$ -connected  $2d$ -dimensional closed manifold where  $d \geq 3$  and let  $n = \text{rank } H_d(M)$ . If  $n > 2$  then the homotopy Lie algebra  $\pi_*^{\mathbb{Q}}(M)$  has trivial center.*

*Proof.* We invoke [15, Proposition 2] which says that a graded Lie algebra  $L$  of finite global dimension has non-trivial center only if the Euler characteristic  $\chi(L)$  is zero, where

$$\chi(L) = \sum_i (-1)^i \dim_{\mathbb{Q}} \text{Ext}_{UL}^i(\mathbb{Q}, \mathbb{Q}),$$

and  $UL$  denotes the universal enveloping algebra of  $L$ . For  $L = \pi_*^{\mathbb{Q}}(M)$ , we have that  $\text{Ext}_{UL}^i(\mathbb{Q}, \mathbb{Q}) \cong H^{id}(M; \mathbb{Q})$ , because  $H^*(M; \mathbb{Q})$  is Koszul dual to  $\pi_*^{\mathbb{Q}}(M)$  (see [8]). It follows that  $L$  has global dimension 2 and that  $\chi(L) = 2 - n$ , whence  $L$  must have trivial center whenever  $n > 2$ .  $\square$

**Lemma 4.10.** *The chain map  $f^*: \text{Der } L \rightarrow \text{Der}_f(\mathbb{L}, L)$  induces an isomorphism in homology in non-negative degrees.*

*Proof.* By inspection, the chain complex  $\text{Der}_f(\mathbb{L}, L)$  is isomorphic to

$$L^n[-d] \xrightarrow{\partial} L[-2d], \quad \partial(\zeta_1, \dots, \zeta_n) = \sum_{i,j} \langle e_i, e_j \rangle [\alpha_i, \zeta_j].$$

Since the intersection form is non-degenerate, it follows that  $\partial$  is surjective (except in degree  $-d-1$ ). The kernel of  $\partial$  is the subspace of  $L^n$  consisting of all elements  $(\zeta_1, \dots, \zeta_n)$  such that

$$\sum_{i,j} \langle e_i, e_j \rangle [\alpha_i, \zeta_j] = 0.$$

This is precisely the condition for  $\theta = \sum \zeta_j \frac{\partial}{\partial \alpha_j}$  to define a derivation on the Lie algebra  $L$ ; the image of the map  $\text{Der } L \rightarrow L^n$ ,  $\theta \mapsto (\theta(\alpha_1), \dots, \theta(\alpha_n))$  is  $\ker \partial$ .  $\square$

## 5. HOMOLOGICAL STABILITY

This section contains the proof of Theorem 1.1 and Theorem 1.5, asserting rational homological stability of the classifying spaces  $\text{Baut}_{\partial}(M_{g,1})$  and  $\widetilde{\text{BDiff}}_{\partial}(M_{g,1})$ .

**5.1. Stabilization maps.** Let  $M_{g,r}$  denote the result of removing the interiors of  $r$  disjointly embedded  $2d$ -disks from the manifold  $M_g = \#^g S^d \times S^d$ . The manifold  $M_{g+1,1}$  may be realized as the union of  $M_{g,1}$  and  $M_{1,2}$  along a common boundary component. An automorphism of  $M_{g,1}$  that fixes the boundary point-wise may be extended to an automorphism of  $M_{g+1,1} = M_{g,1} \cup M_{1,2}$  by letting it act as the identity on  $M_{1,2}$ . This determines a map of monoids  $\sigma: \text{aut}_{\partial}(M_{g,1}) \rightarrow \text{aut}_{\partial}(M_{g+1,1})$ , and hence an induced map on classifying spaces

$$(20) \quad \text{Baut}_{\partial}(M_{g,1}) \rightarrow \text{Baut}_{\partial}(M_{g+1,1}),$$

which we will refer to as the ‘stabilization map’. We need to understand the behavior of the stabilization map in homotopy and homology. Let  $X_g$  denote the space  $\text{Baut}_{\partial}(M_{g,1})$ . Theorem 4.3 yields an isomorphism of graded Lie algebras

$$\pi_*^{\mathbb{Q}}(\widetilde{X}_g) \cong \text{Der}_{\omega_g}^+ \mathbb{L}(H_g^{\mathbb{Q}}).$$

We may choose a basis  $\alpha_1, \beta_1, \dots, \alpha_g, \beta_g$  for  $H_g^{\mathbb{Q}}$  in which

$$\omega_g = [\alpha_1, \beta_1] + \dots + [\alpha_g, \beta_g].$$

There is a ‘stabilization’ morphism  $\text{Der}_{\omega_g}^+ \mathbb{L}(H_g^{\mathbb{Q}}) \rightarrow \text{Der}_{\omega_{g+1}}^+ \mathbb{L}(H_{g+1}^{\mathbb{Q}})$  of Lie algebras defined by extending derivations by zero on the new generators  $\alpha_{g+1}, \beta_{g+1}$ .

**Proposition 5.1.** *The isomorphism*

$$\pi_*^{\mathbb{Q}}(\tilde{X}_g) \cong \text{Der}_{\omega_g}^+ \mathbb{L}(H_g^{\mathbb{Q}})$$

*is compatible with the stabilization maps.*

*Proof.* If  $f$  is a self-equivalence of  $M_{g,1}$ , then  $\sigma(f)$  is the self-map of  $M_{g+1,1}$  that restricts to  $f$  on  $M_{g,1}$  and to the identity on  $M_{1,2}$ , when we realize  $M_{g+1,1}$  as the union of  $M_{g,1}$  and  $M_{1,2}$  along a common boundary component. In other words, the diagram

$$\begin{array}{ccc} & \text{map}_*(M_{g,1}, M_{g+1,1}) & \\ \nearrow i_* & & \uparrow i^* \\ \text{aut}_{\partial}(M_{g,1}) & \xrightarrow{\sigma} & \text{aut}_{\partial}(M_{g+1,1}) \\ \downarrow & & \downarrow j^* \\ * & \xrightarrow{j} & \text{map}_*(M_{1,2}, M_{g+1,1}) \end{array}$$

is commutative. The manifold  $M_{1,2}$  is homotopy equivalent to a wedge of spheres  $S^d \vee S^d \vee S^{2d-1}$ , and a Lie model for it is given by the free graded Lie algebra  $\mathbb{L}(\rho, \alpha, \beta)$  with trivial differential, where the generators  $\alpha, \beta$  have degree  $d-1$ , and  $\rho$  has degree  $2d-2$ . The inclusions  $i$  and  $j$  of  $M_{g,1}$  and  $M_{1,2}$  into  $M_{g+1,1}$  are modeled by the dg Lie algebra morphisms

$$\varphi: \mathbb{L}(H_g^{\mathbb{Q}}) \rightarrow \mathbb{L}(H_{g+1}^{\mathbb{Q}}), \quad \psi: \mathbb{L}(\rho, \alpha, \beta) \rightarrow \mathbb{L}(H_{g+1}^{\mathbb{Q}}),$$

respectively, where  $\psi(\rho) = \omega$ ,  $\psi(\alpha) = \alpha_{g+1}$  and  $\psi(\beta) = \beta_{g+1}$ , and  $\varphi$  is induced by the standard inclusion. From our earlier calculation and the naturality of (13), it follows that the diagram

$$\begin{array}{ccc} & \text{Der}_{\varphi}^+(\mathbb{L}(H_g^{\mathbb{Q}}), \mathbb{L}(H_{g+1}^{\mathbb{Q}})) & \\ \nearrow \varphi_* & & \uparrow \varphi^* \\ \text{Der}^+(\mathbb{L}(H_g^{\mathbb{Q}})) & \xrightarrow{\sigma_*} & \text{Der}^+(\mathbb{L}(H_{g+1}^{\mathbb{Q}})) \\ \downarrow & & \downarrow \psi^* \\ 0 & \xrightarrow{\quad} & \text{Der}_{\psi}^+(\mathbb{L}(\rho, \alpha_{g+1}, \beta_{g+1}), \mathbb{L}(H_{g+1}^{\mathbb{Q}})) \end{array}$$

is commutative. This pins down  $\sigma_*(\theta)$  as the unique derivation on  $\mathbb{L}(H_{g+1}^{\mathbb{Q}})$  that extends  $\theta$  and vanishes on  $\alpha_{g+1}$  and  $\beta_{g+1}$ .  $\square$

**5.2. On the structure of derivation Lie algebras.** In this section we will analyze the structure of the graded Lie algebra  $\text{Der}_{\omega_g} \mathbb{L}(H_g)$  and the Chevalley-Eilenberg complex  $C_*^{CE}(\text{Der}_{\omega_g} \mathbb{L}(H_g))$  as representations of the group  $\Gamma_g$ . The crucial observation, which will allow us to prove homological stability, is that these representations may be identified with the value of certain polynomial functors at the standard representation.

Let  $S^{\epsilon}(\mathbb{Z})$  denote the category of  $\epsilon$ -symmetric inner product spaces over  $\mathbb{Z}$ , for  $\epsilon = \pm 1$ . An object of  $S^{\epsilon}(\mathbb{Z})$  is a finitely generated free  $\mathbb{Z}$ -module  $V$  equipped with a bilinear form  $\langle -, - \rangle_V: V \otimes V \rightarrow \mathbb{Z}$ , which is  $\epsilon$ -symmetric,

$$\langle x, y \rangle_V = \epsilon \langle y, x \rangle_V,$$

and non-singular in the sense that the adjoint map

$$D_V: V \rightarrow V^* = \text{Hom}_{\mathbb{Z}}(V, \mathbb{Z}), \quad x \mapsto \langle x, - \rangle_V,$$

is an isomorphism. A morphism  $f: V \rightarrow W$  in  $S^\epsilon(\mathbb{Z})$  is a linear map such that

$$\langle fx, fy \rangle_W = \langle x, y \rangle_V$$

for all  $x, y \in V$ . The adjoint of  $f$  is the unique linear map  $f^!: W \rightarrow V$  such that

$$\langle f^!x, y \rangle_V = \langle x, fy \rangle_W$$

for all  $x \in W, y \in V$ . Clearly  $f^!f = 1_V$ . In particular, the linear map  $f: V \rightarrow W$  is injective and there is an isomorphism of inner product spaces

$$W \cong V \oplus V^\perp, \quad x \mapsto (f^!(x), x - ff^!(x)),$$

where

$$V^\perp = \{x \in W \mid \langle x, fy \rangle_W = 0 \text{ for all } y \in V\} = \ker f^!.$$

Fix an integer  $d > 1$  and let  $\epsilon = (-1)^d$ . We will describe a functor

$$S^\epsilon(\mathbb{Z}) \rightarrow (\text{graded Lie algebras over } \mathbb{Z}), \quad V \mapsto \text{Der}_\omega \mathbb{L}(V).$$

Given a  $\mathbb{Z}$ -module  $V$ , let  $\mathbb{L}(V)$  denote the free graded Lie algebra over  $\mathbb{Z}$  generated by the graded  $\mathbb{Z}$ -module  $V[d-1]$ , i.e., we put  $V$  in homological degree  $d-1$ . We will not distinguish notationally between elements of  $V$  and their images in  $\mathbb{L}(V)$ . Given a linear map  $f: V \rightarrow W$  there is an induced morphism of graded Lie algebras  $\mathbb{L}(f): \mathbb{L}(V) \rightarrow \mathbb{L}(W)$ . Given a morphism  $f: V \rightarrow W$  in  $S^\epsilon(\mathbb{Z})$ , we define a morphism of Lie algebras

$$\chi_f: \text{Der } \mathbb{L}(V) \rightarrow \text{Der } \mathbb{L}(W)$$

as follows. For  $\theta \in \text{Der } \mathbb{L}(V)$ , we let  $\chi_f(\theta) \in \text{Der } \mathbb{L}(W)$  be the unique derivation that satisfies

$$\chi_f(\theta)(x) = \mathbb{L}(f)\theta(f^!x)$$

for all  $x \in W$ . It is easy to check that  $\chi_g\chi_f = \chi_{gf}$  when the composition  $gf$  is defined. It is perhaps not evident from the definition that  $\chi_f$  is a morphism of Lie algebras, but we will verify this below.

**Proposition 5.2.** *Let  $f: V \rightarrow W$  be a morphism in  $S^\epsilon(\mathbb{Z})$ . Then  $\chi_f: \text{Der } \mathbb{L}(V) \rightarrow \text{Der } \mathbb{L}(W)$  is an injective morphism of graded Lie algebras.*

*Proof.* Let  $\theta, \eta \in \text{Der } \mathbb{L}(V)$ . First note that we have the following equality of maps from  $\mathbb{L}(V)$  to  $\mathbb{L}(W)$ .

$$(21) \quad \chi_f(\theta) \circ \mathbb{L}(f) = \mathbb{L}(f) \circ \theta.$$

This follows because both sides are  $\mathbb{L}(f)$ -derivations and for every  $y \in V$  we have

$$\chi_f(\theta)\mathbb{L}(f)(y) = \chi_f(\theta)(fy) = \mathbb{L}(f)\theta(f^!fy) = \mathbb{L}(f)(y).$$

Next, to verify the equality

$$[\chi_f(\theta), \chi_f(\eta)] = \chi_f[\theta, \eta],$$

note that both sides are derivations, so equality may be checked by evaluating at generators  $x \in V$ . But

$$\begin{aligned} [\chi_f(\theta), \chi_f(\eta)](x) &= \chi_f(\theta)\chi_f(\eta)(x) - (-1)^{|\chi_f(\theta)||\chi_f(\eta)|}\chi_f(\eta)\chi_f(\theta)(x) \\ &= \chi_f(\theta)\mathbb{L}(f)\eta(f^!x) - (-1)^{|\theta||\eta|}\chi_f(\eta)\mathbb{L}(f)\theta(f^!x) \\ &= \mathbb{L}(f)\theta\eta(f^!x) - (-1)^{|\theta||\eta|}\eta\theta(f^!x) \\ &= \chi_f([\theta, \eta])(x), \end{aligned}$$

where we have used (21) in the middle step.

To check injectivity, define the map

$$\psi_f: \text{Der } \mathbb{L}(W) \rightarrow \text{Der } \mathbb{L}(V)$$

by requiring

$$\psi_f(\theta)(x) = \mathbb{L}(f^!) \theta(fx),$$

for  $\theta \in \text{Der } \mathbb{L}(W)$  and  $x \in V$ . Then one easily checks that the composite  $\psi_f \circ \theta_f$  is the identity map on  $\text{Der } \mathbb{L}(V)$ . (Note however that  $\psi_f$  is not necessarily a morphism of Lie algebras.)  $\square$

Let  $\Lambda(V)$  denote the free graded commutative algebra over  $\mathbb{Z}$  on the graded  $\mathbb{Z}$ -module  $V[d]$ . Thus,  $\Lambda(V)$  is an exterior algebra if  $d$  is odd and a polynomial algebra if  $d$  is even. The bilinear form  $V \otimes V \rightarrow \mathbb{Z}$  extends to a bilinear form

$$\langle -, - \rangle: \mathbb{L}^2(V) \otimes \Lambda^2(V) \rightarrow \mathbb{Z}$$

given by

$$\langle [x, y], a \wedge b \rangle = \langle x, a \rangle \langle y, b \rangle + \epsilon \langle y, a \rangle \langle x, b \rangle.$$

The adjoint map

$$\mathbb{L}^2(V) \rightarrow \text{Hom}_{\mathbb{Z}}(\Lambda^2(V), \mathbb{Z})$$

is an isomorphism. Thus, there is a unique element  $\omega_V \in \mathbb{L}^2(V)$  that satisfies

$$\langle \omega_V, a \wedge b \rangle = \langle a, b \rangle_V$$

for all  $a, b \in V$ . If we choose a basis  $e_1, \dots, e_n$  for  $V$ , we have the dual basis  $e_1^\#, \dots, e_n^\#$ , uniquely determined by

$$\langle e_i, e_j^\# \rangle = \delta_{ij}.$$

Then

$$e_i^\# = \sum_{j=1}^n \langle e_i^\#, e_j^\# \rangle e_j$$

and it is easy to check that

$$2\omega_V = \sum_{i,j} \langle e_i^\#, e_j^\# \rangle [e_i, e_j] = \sum_{i=1}^n [e_i, e_i^\#].$$

As an example, the hyperbolic module  $H_g$  has a basis  $e_1, \dots, e_g, f_1, \dots, f_g$  where

$$\langle e_i, e_j \rangle = 0, \quad \langle e_i, f_j \rangle = \delta_{ij}, \quad \langle f_i, f_j \rangle = 0,$$

and the element  $\omega_g = \omega_{H_g}$  has the form

$$\omega_g = [e_1, f_1] + \dots + [e_g, f_g].$$

The obvious inclusion  $H_g \rightarrow H_{g+1}$  is a morphism in  $S^\epsilon(\mathbb{Z})$ , and the induced morphism of graded Lie algebras

$$\text{Der } \mathbb{L}(H_g) \rightarrow \text{Der } \mathbb{L}(H_{g+1})$$

is given by the extending a derivation by zero on the new generators  $e_{g+1}$  and  $f_{g+1}$ .

In general, evaluation at  $\omega_V \in \mathbb{L}^2(V)$  defines a map

$$ev_{\omega_V}: \text{Der } \mathbb{L}(V) \rightarrow \mathbb{L}(V), \quad \theta \mapsto \theta(\omega_V).$$

Explicitly, if  $\theta$  a derivation on  $\mathbb{L}(V)$ , then

$$\theta(\omega_V) = \sum_{i,j} \langle e_i^\#, e_j^\# \rangle [\theta e_i, e_j].$$

**Proposition 5.3.** *Given a morphism  $f: V \rightarrow W$  in  $S^\epsilon(\mathbb{Z})$ , the diagram*

$$\begin{array}{ccc} \mathrm{Der} \mathbb{L}(V) & \xrightarrow{ev_{\omega_V}} & \mathbb{L}(V) \\ \chi_f \downarrow & & \downarrow \mathbb{L}(f) \\ \mathrm{Der} \mathbb{L}(W) & \xrightarrow{ev_{\omega_W}} & \mathbb{L}(W) \end{array}$$

*is commutative.*

*Proof.* As we noted above, the map  $f: V \rightarrow W$  is injective and induces an isomorphism of inner product spaces

$$W \cong V \oplus V^\perp,$$

so we may without loss of generality assume that  $W = V \oplus V^\perp$ . Then we have that  $\omega_W = \omega_V + \omega_{V^\perp}$ , where  $\omega_V \in \mathbb{L}(V)$  and  $\omega_{V^\perp} \in \mathbb{L}(V^\perp)$ . For a derivation  $\theta$  on  $\mathbb{L}(V)$ , we may describe  $\chi_f(\theta)$  as the unique derivation on  $\mathbb{L}(W)$  that restricts to  $\theta$  on  $\mathbb{L}(V)$  and restricts to zero on  $\mathbb{L}(V^\perp)$ . Thus,

$$\chi_f(\theta)(\omega_W) = \chi_f(\theta)(\omega_V) + \chi_f(\theta)(\omega_{V^\perp}) = \theta(\omega_V),$$

which proves the claim.  $\square$

The kernel of the evaluation map will be denoted  $\mathrm{Der}_{\omega_V} \mathbb{L}(V)$ . It is a graded Lie subalgebra of  $\mathrm{Der} \mathbb{L}(V)$ . By definition, we have an exact sequence

$$0 \rightarrow \mathrm{Der}_{\omega_V} \mathbb{L}(V) \rightarrow \mathrm{Der} \mathbb{L}(V) \rightarrow \mathbb{L}(V),$$

and by the previous proposition this is natural for maps in  $S^\epsilon(\mathbb{Z})$ .

For  $x \in V$  and  $\xi \in \mathbb{L}(V)$  we may specify a derivation  $\theta_{x,\xi}$  on  $\mathbb{L}(V)$  by declaring that

$$\theta_{x,\xi}(y) = (-1)^{(|\xi|-1)(d-1)} \langle x, y \rangle_V \xi$$

for  $y \in V$ . This defines a map

$$\theta_{-, -}: V \otimes \mathbb{L}(V) \rightarrow \mathrm{Der} \mathbb{L}(V).$$

In fact, since the inner product is non-singular and since a derivation is determined by its value on generators, the above map is an isomorphism. Moreover, the map is natural with respect to morphisms  $f: V \rightarrow W$  in  $S^\epsilon(\mathbb{Z})$  in the sense that the diagram

$$\begin{array}{ccc} V \otimes \mathbb{L}(V) & \xrightarrow{\theta_{-, -}} & \mathrm{Der} \mathbb{L}(V) \\ f \otimes \mathbb{L}(f) \downarrow & & \downarrow \chi_f \\ W \otimes \mathbb{L}(W) & \xrightarrow{\theta_{-, -}} & \mathrm{Der} \mathbb{L}(W) \end{array}$$

is commutative. Indeed, for  $x \in V$ ,  $y \in W$  and  $\xi \in \mathbb{L}(V)$ , we have

$$\chi_f(\theta_{x,\xi})(y) = \mathbb{L}(f)\theta_{x,\xi}(f^!y) = (-1)^{(|\xi|-1)(d-1)} \mathbb{L}(f)\langle x, f^!y \rangle \xi,$$

and on the other hand

$$\theta_{fx, \mathbb{L}(f)\xi}(y) = (-1)^{(|\xi|-1)(d-1)} \langle fx, y \rangle \mathbb{L}(f)\xi = (-1)^{(|\xi|-1)(d-1)} \mathbb{L}(f)\langle x, f^!y \rangle \xi.$$

Next, we will interpret the evaluation map in terms of this isomorphism.

**Proposition 5.4.** *The diagram*

$$\begin{array}{ccc} V \otimes \mathbb{L}(V) & \xrightarrow{[-, -]} & \mathbb{L}(V) \\ \theta_{-, -} \downarrow & & \parallel \\ \mathrm{Der} \mathbb{L}(V) & \xrightarrow{ev_{\omega_V}} & \mathbb{L}(V) \end{array}$$

is commutative.

*Proof.* Choose a basis  $e_1, \dots, e_n$  for  $V$  and let  $e_1^\#, \dots, e_n^\#$  be the dual basis. Every element  $x \in V$  may be written in these bases as

$$x = \sum_{i=1}^n \langle x, e_i^\# \rangle e_i = \sum_{i=1}^n (-1)^d \langle x, e_i \rangle e_i^\#.$$

By using these relations and the graded anti-symmetry of the Lie bracket we obtain

$$\begin{aligned} \theta_{x,\xi}(\omega_V) &= \sum_{i,j} \langle e_i^\#, e_j^\# \rangle [\theta_{x,\xi}(e_i), e_j] \\ &= \sum_{i,j} (-1)^{(|\xi|-1)(d-1)} \langle e_i^\#, e_j^\# \rangle [\langle x, e_i \rangle \xi, e_j] \\ &= \sum_j (-1)^{(|\xi|-1)(d-1)+d} \langle x, e_j^\# \rangle [\xi, e_j] \\ &= (-1)^{(|\xi|-1)(d-1)+d} [\xi, x] \\ &= [x, \xi]. \end{aligned}$$

□

Let  $\mathfrak{g}(V)$  be the kernel of the map  $V \otimes \mathbb{L}(V) \rightarrow \mathbb{L}(V)$ . It follows that we have a commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathfrak{g}(V) & \longrightarrow & V \otimes \mathbb{L}(V) & \xrightarrow{[-,-]} & \mathbb{L}(V) \\ & & \downarrow \cong & & \downarrow \theta_{-,-} \cong & & \parallel \\ 0 & \longrightarrow & \text{Der}_{\omega_V} \mathbb{L}(V) & \longrightarrow & \text{Der } \mathbb{L}(V) & \xrightarrow{ev_{\omega_V}} & \mathbb{L}(V) \end{array}$$

All maps in the above diagram are natural with respect to morphisms  $f: V \rightarrow W$  in  $S^\epsilon(\mathbb{Z})$ , so we obtain a natural isomorphism

$$\text{Der}_{\omega_V} \mathbb{L}(V) \cong \mathfrak{g}(V).$$

The interesting thing to notice here is that the top row is functorial not only with respect to morphisms in  $S^\epsilon(\mathbb{Z})$ , but for all linear maps between abelian groups. In fact more can be said; we may identify the top row as the values at  $V$  of certain Schur functors.

Let  $\mathcal{L}ie = \{\mathcal{L}ie(k)\}_{k \geq 0}$  denote the Lie operad, and choose a generator  $\lambda$  for  $\mathcal{L}ie(2)$  (we refer the reader to [32] for basic facts about operads). For every  $k \geq 2$  the map

$$\mathcal{L}ie(k-1) \rightarrow \mathcal{L}ie(k), \quad \gamma \mapsto \lambda \circ_2 \gamma$$

is  $\Sigma_{k-1}$ -equivariant, where the action on  $\mathcal{L}ie(k)$  comes from viewing  $\Sigma_{k-1}$  as the isotropy subgroup of 1 in  $\Sigma_k$ . Let  $\mathcal{U}(k)$  denote the kernel of the adjoint map. Thus, there is a short exact sequence of  $\Sigma_k$ -representations

$$(22) \quad 0 \rightarrow \mathcal{U}(k) \rightarrow \text{Ind}_{\Sigma_{k-1}}^{\Sigma_k} \mathcal{L}ie(k-1) \rightarrow \mathcal{L}ie(k) \rightarrow 0.$$

If we apply the functor  $- \otimes_{\Sigma_k} V^{\otimes k}$  to (22), for  $V$  any abelian group, we obtain an exact sequence

$$\text{Tor}_1^{\mathbb{Z}[\Sigma_k]}(\mathcal{L}ie(k), V^{\otimes k}) \rightarrow \mathcal{U}(k) \otimes_{\Sigma_k} V^{\otimes k} \rightarrow V \otimes \mathcal{L}ie^{k-1}(V) \rightarrow \mathcal{L}ie^k(V) \rightarrow 0,$$

where the last map may be identified with the bracketing map

$$V \otimes \mathcal{L}ie^{k-1}(V) \rightarrow \mathcal{L}ie^k(V), \quad x \otimes \xi \mapsto [x, \xi].$$

In fact, the Tor-group  $\text{Tor}_1^{\mathbb{Z}[\Sigma_k]}(\mathcal{L}ie(k), V^{\otimes k})$  vanishes. This is non-trivial, but it follows from the results in [3] using techniques similar to those in [2]. We will not

give a proof here, because for our main application we only need the result over  $\mathbb{Q}$ , where it is obvious. In any event, the kernel  $\mathfrak{g}^k(V)$  of the bracketing map may be naturally identified with

$$\mathfrak{g}^k(V) \cong \mathcal{U}(k) \otimes_{\Sigma_k} V^{\otimes k}.$$

The above considerations imply the following result.

**Proposition 5.5.** *Let  $V$  be an  $\epsilon$ -symmetric inner product space over  $\mathbb{Z}$ . There is a natural isomorphism of graded  $\mathbb{Z}$ -modules*

$$\mathrm{Der}_{\omega_V}^+ \mathbb{L}(V) \cong \bigoplus_{k \geq 3} \widetilde{\mathcal{U}}(k) \otimes_{\Sigma_k} V^{\otimes k},$$

where the graded  $\Sigma_k$ -representation  $\widetilde{\mathcal{U}}(k)$  is described by the following:  $\widetilde{\mathcal{U}}(k)$  is concentrated in degree  $(k-2)(d-1)$  where it is equal to  $\mathcal{U}(k)$  if  $d$  is odd, and to  $\mathcal{U}(k)$  twisted by the sign representation if  $d$  is even. The  $\Sigma_k$ -module  $\mathcal{U}(k)$  is defined by (22).

We will now proceed to a Schur functor description of the Chevalley-Eilenberg chains on  $\mathrm{Der}_{\omega_V}^+ \mathbb{L}(V)$ .

**Proposition 5.6.** *Let  $d \geq 2$  and let  $V$  be an  $\epsilon$ -symmetric inner product space over  $\mathbb{Z}$ . There is a natural isomorphism of graded  $\mathbb{Z}$ -modules*

$$C_*^{CE}(\mathrm{Der}_{\omega_V}^+ \mathbb{L}(V)) \cong \bigoplus_{k \geq 0} \mathcal{C}(k) \otimes_{\Sigma_k} V^{\otimes k},$$

where the graded  $\mathbb{Z}[\Sigma_k]$ -module  $\mathcal{C}(k)$  is concentrated in degrees  $\geq \frac{kd}{3}$ .

*Proof.* As a graded  $\mathbb{Z}$ -module, the Chevalley-Eilenberg chains on a graded Lie algebra  $L$  may be described as the value at  $L$  of a Schur functor:

$$C_*^{CE}(L) = \bigoplus_{k \geq 0} \Lambda(k) \otimes_{\Sigma_k} L^{\otimes k},$$

where  $\Lambda(k)$  is the trivial representation of  $\Sigma_k$  concentrated in degree  $k$ . It follows from Proposition 5.5 that  $C_*^{CE}(\mathrm{Der}_{\omega_V}^+ \mathbb{L}(V))$  is the value at  $V$  of the composite Schur functor  $\mathcal{C}(-) = \Lambda \circ \widetilde{\mathcal{U}}(-)$ , i.e.,

$$C_*^{CE}(\mathrm{Der}_{\omega_V}^+ \mathbb{L}(V)) \cong \bigoplus_{k \geq 0} \mathcal{C}(k) \otimes_{\Sigma_k} V^{\otimes k}.$$

The composite Schur functor  $\mathcal{C} = \Lambda \circ \widetilde{\mathcal{U}}$  is described by

$$(23) \quad \mathcal{C}(k) = \bigoplus_{r \geq 0} \Lambda(r) \otimes_{\Sigma_r} \left( \bigoplus_{\Sigma_{i_1} \times \dots \times \Sigma_{i_r}}^{\Sigma_k} (\widetilde{\mathcal{U}}(i_1) \otimes \dots \otimes \widetilde{\mathcal{U}}(i_r)) \right),$$

where the second sum is over all  $r$ -tuples  $(i_1, \dots, i_r)$  of non-negative integers such that  $i_1 + \dots + i_r = k$ .

Fix  $r$  and  $(i_1, \dots, i_r)$ , and consider the corresponding summand of (23). If the summand is non-zero, then we must have that  $i_j \geq 3$  for every  $j$ , because  $\widetilde{\mathcal{U}}(i_j) = 0$  if  $i_j < 3$ . But then  $k = i_1 + \dots + i_r \geq 3r$ . Therefore, the summand in question is concentrated in degree

$$\begin{aligned} r + (i_1 - 2)(d - 1) + \dots + (i_r - 2)(d - 1) &= r + (k - 2r)(d - 1) \\ &= r(3 - 2d) + k(d - 1) \\ &\geq \frac{k}{3}(3 - 2d) + k(d - 1) \\ &= \frac{kd}{3}. \end{aligned}$$



where the inequality follows because  $d \geq 2$  so that  $3 - 2d$  is negative. Thus,  $\mathcal{C}(k) = (\Lambda \circ \widetilde{\mathcal{U}})(k)$  is concentrated in degrees  $\geq \frac{kd}{3}$ .  $\square$

**5.3. Polynomial functors and homological stability.** We adopt a naive approach to polynomial functors. By a *polynomial functor of degree  $\leq \ell$*  we will mean a functor  $P$  from abelian groups to itself isomorphic to a functor of the form

$$P(V) = \bigoplus_{k=0}^{\ell} P(k) \otimes_{\Sigma_k} V^{\otimes k},$$

for some sequence of abelian groups  $P(k)$  with an action of the symmetric group  $\Sigma_k$ . Recall that  $\Gamma_g$  denotes the automorphism group of the hyperbolic quadratic module  $(H_g, \mu, q)$ .

**Theorem 5.7** (Charney [17, Theorem 4.3]). *Let  $P$  be a polynomial functor of degree  $\leq \ell$ . Then the stabilization map*

$$H_k(\Gamma_g; P(H_g)) \rightarrow H_k(\Gamma_{g+1}; P(H_{g+1}))$$

*is an isomorphism for  $g > 2k + \ell + 4$  and a surjection for  $g = 2k + \ell + 4$ .*

**Remark 5.8.** Charney's theorem is in fact more general. In the notation of [17], the group  $\Gamma_g$  is isomorphic to the automorphism group of the hyperbolic module in the category  $Q^\lambda(A, \Lambda)$ , where  $A$  is the ring  $\mathbb{Z}$  with trivial involution,  $\lambda = (-1)^d$ , and  $\Lambda = 0$  if  $d$  is even,  $\Lambda = \mathbb{Z}$  if  $d = 1, 3, 7$ , and  $\Lambda = 2\mathbb{Z}$  if  $d \neq 1, 3, 7$  is odd. Also, one can replace  $P(H_g)$  by a 'central coefficient system of degree  $\leq \ell$ '. It is easy to verify that  $P(H_g)$  is a central coefficient system of degree  $\leq \ell$  whenever  $P$  is a polynomial functor of degree  $\leq \ell$ .

Let  $\mathfrak{g}_g$  denote the graded Lie algebra

$$\mathfrak{g}_g = \text{Der}_{\omega_g}^+ \mathbb{L}(H_g)$$

with its natural  $\Gamma_g$ -action. Let  $\sigma = \chi_f: \mathfrak{g}_g \rightarrow \mathfrak{g}_{g+1}$  be the morphism of graded Lie algebras induced by the inclusion  $H_g \rightarrow H_{g+1}$ .

**Proposition 5.9.** *Let  $d \geq 2$  and fix a non-negative integer  $p$ . The map*

$$\sigma_*: H_q(\Gamma_g; C_p^{CE}(\mathfrak{g}_g)) \rightarrow H_q(\Gamma_{g+1}; C_p^{CE}(\mathfrak{g}_{g+1}))$$

*is an isomorphism for  $g > 2q + \lfloor \frac{3p}{d} \rfloor + 4$  and a surjection for  $g = 2q + \lfloor \frac{3p}{d} \rfloor + 4$ .*

*Proof.* According to Proposition 5.6, there is an isomorphism

$$C_p^{CE}(\mathfrak{g}_g) = \bigoplus_{k \geq 0} \mathcal{C}(k)_p \otimes_{\Sigma_k} H_g^{\otimes k},$$

where  $\mathcal{C}(k)_p = 0$  unless  $p \geq \frac{kd}{3}$ , i.e.,  $k \leq \frac{3p}{d}$ . Thus,  $C_p^{CE}(\mathfrak{g}_g)$  may be identified with the value at  $H_g$  of a polynomial functor of degree  $\leq \lfloor \frac{3p}{d} \rfloor$ . The claim then follows from Charney's theorem.  $\square$

**Theorem 5.10.** *Let  $d \geq 2$ . The map in hyperhomology*

$$\sigma_*: \mathbb{H}_k(\Gamma_g; C_*^{CE}(\mathfrak{g}_g)) \rightarrow \mathbb{H}_k(\Gamma_{g+1}; C_*^{CE}(\mathfrak{g}_{g+1}))$$

*is an isomorphism for  $g > 2k + 4$  and surjective for  $g = 2k + 4$ .*

*Proof.* Consider the first page of the first hyperhomology spectral sequence

$${}^I E_{p,q}^1(g) = H_q(\Gamma_g; C_p^{CE}(\mathfrak{g}_g)) \Rightarrow \mathbb{H}_{p+q}(\Gamma_g; C_*^{CE}(\mathfrak{g}_g)).$$

The map  ${}^I E_{p,q}^1(g) \rightarrow {}^I E_{p,q}^1(g+1)$  is an isomorphism for  $n > 2q + 2p + 4$  and a surjection for  $n = 2q + 2p + 4$  by Proposition 5.9, because  $\frac{3p}{d} \leq 2p$  when  $d \geq 2$ . The claim then follows from the mapping theorem for spectral sequences.  $\square$

So far we have worked with coefficients in  $\mathbb{Z}$ . When we extend scalars to  $\mathbb{Q}$ , a vanishing theorem of Borel allows us to calculate the stable cohomology in terms of cohomology with trivial coefficients and invariants.

**Theorem 5.11.** *If  $P$  is a polynomial functor of degree  $\leq \ell$ , then the natural map*

$$H^k(\Gamma_g; \mathbb{Q}) \otimes P(H_g^{\mathbb{Q}})^{\Gamma_g} \rightarrow H^k(\Gamma_g; P(H_g^{\mathbb{Q}}))$$

*is an isomorphism for  $g > 2k + \ell + 4$ .*

*Proof.* This follows by combining Charney's theorem (Theorem 5.7) with Borel's vanishing theorem [13, Theorem 4.4]. The group  $\Gamma_g$  is an arithmetic subgroup of the algebraic group  $\mathrm{Sp}_g$  or  $O_{g,g}$ , depending on whether  $d$  is odd or even. Call this algebraic group  $G_g$ . If  $P$  is a polynomial functor, then  $P(H_g^{\mathbb{Q}})$  is a rational representation of the algebraic group  $G_g$ , and we may decompose it as a direct sum,

$$P(H_g^{\mathbb{Q}}) = P(H_g^{\mathbb{Q}})^{G_g} \oplus E_1^g \oplus \cdots \oplus E_{r_g}^g,$$

where  $E_1^g, \dots, E_{r_g}^g$  are the non-trivial irreducible subrepresentations. It is easy to check that because  $P$  is polynomial of degree  $\leq \ell$ , the coefficients of the highest weight of  $E_i^g$  are bounded above by  $\ell$ , for all  $g$  and all  $i$ . As explained in [13, §4.6], this implies that for every  $k$ , there is an  $n(k)$  such that

$$H^k(\Gamma_g; E_i^g) = 0, \quad \text{for all } g \geq n(k) \text{ and all } i.$$

It follows that the map induced by the inclusion of  $P(H_g^{\mathbb{Q}})^{G_g}$  into  $P(H_g^{\mathbb{Q}})$ ,

$$H^k(\Gamma_g; \mathbb{Q}) \otimes P(H_g^{\mathbb{Q}})^{G_g} \cong H^k(\Gamma_g; P(H_g^{\mathbb{Q}})^{G_g}) \rightarrow H^k(\Gamma_g; P(H_g^{\mathbb{Q}})),$$

is an isomorphism for all  $g \geq n(k)$ . Thus, for  $k$  fixed, the vertical maps in the diagram

$$\begin{array}{ccccc} H^k(\Gamma_g; \mathbb{Q}) \otimes P(H_g^{\mathbb{Q}})^{G_g} & \longrightarrow & H^k(\Gamma_{g+1}; \mathbb{Q}) \otimes P(H_{g+1}^{\mathbb{Q}})^{G_{g+1}} & \longrightarrow & \cdots \\ \downarrow & & \downarrow & & \\ H^k(\Gamma_g; P(H_g^{\mathbb{Q}})) & \longrightarrow & H^k(\Gamma_{g+1}; P(H_{g+1}^{\mathbb{Q}})) & \longrightarrow & \cdots \end{array}$$

will start to become isomorphisms after continuing far enough to the right. But both the top and the bottom horizontal maps are isomorphisms for  $g > 2k + \ell + 4$  by Theorem 5.7, so the vertical maps had better be isomorphisms already for  $g > 2k + \ell + 4$ , no matter what  $n(k)$  is. Finally, we should point out that  $V^{\Gamma_g} = V^{G_g}$  for any rational representation  $V$  because of density of  $\Gamma_g$  in  $G_g$  (see e.g. [11]).  $\square$

**5.4. Homological stability for homotopy automorphisms.** In this section we will prove the following theorem:

**Theorem 5.12.** *Let  $d \geq 3$ . The map*

$$\sigma_*: H_k(\mathrm{Baut}_{\partial}(M_{g,1}); \mathbb{Q}) \rightarrow H_k(\mathrm{Baut}_{\partial}(M_{g+1,1}); \mathbb{Q})$$

*is an isomorphism for  $g > 2k + 4$  and a surjection for  $g = 2k + 4$ .*

Recall that  $M_{g,1}$  denotes the manifold  $\#^g S^d \times S^d$  with an open  $2d$ -disk removed. Throughout this section we will use the notation

$$\begin{aligned} X_g &= \mathrm{Baut}_{\partial}(M_{g,1}), \\ \Gamma_g &= \mathrm{Aut}(H_g, \mu, q), \\ \mathfrak{g}_g &= \pi_*^{\mathbb{Q}}(\tilde{X}_g) \cong \mathrm{Der}_{\omega_g}^+ \mathbb{L}(H_g^{\mathbb{Q}}). \end{aligned}$$

The universal cover spectral sequence,

$$E_{p,q}^2 = H_p(\pi_1(X); H_q(\tilde{X})) \Rightarrow H_{p+q}(X),$$

is natural in  $X$ . To prove Theorem 5.12 it is therefore sufficient to show that

$$\sigma: H_p(\pi_1(X_g); H_q(\tilde{X}_g; \mathbb{Q})) \rightarrow H_p(\pi_1(X_{g+1}); H_q(\tilde{X}_{g+1}; \mathbb{Q}))$$

is an isomorphism if  $g > 2p + 2q + 4$  and a surjection for  $g = 2p + 2q + 4$ . This will follow from Proposition 5.14 and Proposition 5.15 below.

**Proposition 5.13.** *The group  $\pi_1(X_g)$  is rationally perfect for  $g \geq 2$ .*

*Proof.* By Proposition 4.1, there is a short exact sequence of groups,

$$1 \rightarrow K_g \rightarrow \pi_1(X_g) \rightarrow \Gamma_g \rightarrow 1,$$

where the kernel  $K_g$  is finite, whence rationally perfect. The group  $\Gamma_g$  is an arithmetic subgroup of the algebraic group  $\mathrm{Sp}_g$  or  $O_{g,g}$ , depending on whether  $d$  is odd or even. In either case, the algebraic group is almost simple and its  $\mathbb{Q}$ -rank is  $g$ . Hence, it follows from Theorem A.1 that  $\Gamma_g$  is rationally perfect. An application of the Hochschild-Serre spectral sequence then shows that  $\pi_1(X_g)$  is rationally perfect.  $\square$

**Proposition 5.14.** *For  $d \geq 3$ ,  $g \geq 2$ , and all  $p, q$ , there is an isomorphism*

$$H_p(\pi_1(X_g); H_q(\tilde{X}_g; \mathbb{Q})) \cong H_p(\Gamma_g; H_q^{CE}(\mathfrak{g}_g)),$$

*compatible with the stabilization maps.*

*Proof.* By combining Theorem 4.3, Proposition 5.13 and Proposition 2.3, it follows that there is a  $\pi_1(X_g)$ -equivariant isomorphism

$$H_*(\tilde{X}_g; \mathbb{Q}) \cong H_*^{CE}(\mathfrak{g}_g).$$

Compatibility with the stabilization maps follows from Proposition 5.1 and naturality of the Quillen spectral sequence. By Theorem 4.3, the kernel of the homomorphism  $\pi_1(X_g) \rightarrow \Gamma_g$  is a finite group that acts trivially on  $\mathfrak{g}_g$  and hence on  $H_q^{CE}(\mathfrak{g}_g)$ . It is easy to see that the homomorphism  $\pi_1(X_g) \rightarrow \Gamma_g$  is compatible with the stabilization maps. This implies that there is a compatible isomorphism

$$H_p(\pi_1(X_g); H_q(\tilde{X}_g; \mathbb{Q})) \cong H_p(\Gamma_g; H_q^{CE}(\mathfrak{g}_g)),$$

as claimed.  $\square$

**Proposition 5.15.** *Let  $d \geq 2$ . The stabilization map*

$$\sigma_*: H_p(\Gamma_g; H_q^{CE}(\mathfrak{g}_g)) \rightarrow H_p(\Gamma_{g+1}; H_q^{CE}(\mathfrak{g}_{g+1}))$$

*is an isomorphism for  $g > 2p + 2q + 4$  and a surjection for  $g = 2p + 2q + 4$ .*

*Proof.* For  $g \geq 2$  the group  $\Gamma_g$  is rationally perfect, see Theorem A.1. The chain complex of  $\mathbb{Q}[\Gamma_g]$ -modules  $C_*^{CE}(\mathfrak{g}_g)$ , being finite dimensional over  $\mathbb{Q}$  in each degree, is therefore split by Proposition B.5. By Lemma B.1 we get a homotopy commutative diagram of chain complexes of  $\mathbb{Q}[\Gamma_g]$ -modules

$$\begin{array}{ccc} C_*^{CE}(\mathfrak{g}_g) & \xrightarrow{\sigma} & C_*^{CE}(\mathfrak{g}_{g+1}) \\ \simeq \downarrow & & \downarrow \simeq \\ H_*^{CE}(\mathfrak{g}_g) & \xrightarrow{\sigma} & H_*^{CE}(\mathfrak{g}_{g+1}) \end{array}$$

where the vertical maps are chain homotopy equivalences. This implies that the diagram

$$\begin{array}{ccc} \mathbb{H}_k(\Gamma_g; C_*^{CE}(\mathfrak{g}_g)) & \xrightarrow{\sigma} & \mathbb{H}_k(\Gamma_{g+1}; C_*^{CE}(\mathfrak{g}_{g+1})) \\ \cong \downarrow & & \downarrow \cong \\ \mathbb{H}_k(\Gamma_g; H_*^{CE}(\mathfrak{g}_g)) & \xrightarrow{\sigma} & \mathbb{H}_k(\Gamma_{g+1}; H_*^{CE}(\mathfrak{g}_{g+1})) \end{array}$$

is commutative, and that the vertical maps are isomorphisms. By Theorem 5.10, the top map is an isomorphism for  $g > 2k + 4$  and a surjection for  $g \geq 2k + 4$ , so the same is true of the bottom map. For any group  $\Gamma$  and any graded  $\Gamma$ -module  $H_*$ , regarded as a chain complex with zero differential, there is a decomposition

$$\mathbb{H}_k(\Gamma; H_*) \cong \bigoplus_{p+q=k} H_p(\Gamma; H_q),$$

which is natural in  $\Gamma$  and  $H_*$ . It follows that the constituents of the bottom map,

$$\sigma_{p,q}: H_p(\Gamma_g; H_q^{CE}(\mathfrak{g}_g)) \rightarrow H_p(\Gamma_{g+1}; H_q^{CE}(\mathfrak{g}_{g+1})),$$

are isomorphisms for  $g > 2p + 2q + 4$  and surjections for  $g = 2p + 2q + 4$ .  $\square$

**Remark 5.16.** If  $C_*$  is a split chain complex of  $\Gamma$ -modules, then  $H_p(\Gamma; H_q(C_*))$  is isomorphic to the  $q$ -th homology of the chain complex

$$\cdots \rightarrow H_p(\Gamma; C_{q+1}) \rightarrow H_p(\Gamma; C_q) \rightarrow H_p(\Gamma; C_{q-1}) \rightarrow \cdots.$$

Indeed, the functor  $H_p(\Gamma; -)$  from  $\Gamma$ -modules to abelian groups is additive, so it preserves chain homotopy equivalences.

**5.5. Homological stability for block diffeomorphisms.** Recall that the group of block diffeomorphisms  $\widetilde{\text{Diff}}(M)$  of a manifold  $M$  is defined as the geometric realization of the semi-simplicial group whose  $k$ -simplices are the self-diffeomorphisms  $\varphi$  of  $\Delta^k \times M$  that preserve faces, in the sense that  $\varphi(\partial_i \Delta^k \times M) \subseteq \partial_i \Delta^k \times M$  for all  $i$ . The geometric realization of the semi-simplicial subgroup of diffeomorphisms of  $\Delta^k \times M$  over  $\Delta^k$  is homotopy equivalent to the diffeomorphism group  $\text{Diff}(M)$ , so the block diffeomorphism group may be viewed as an enlargement of the diffeomorphism group (see, e.g., [9] for more details). If  $M$  has a non-empty boundary, then we may consider the group of boundary preserving block diffeomorphisms,  $\widetilde{\text{Diff}}_\partial(M)$ , which is defined as above but with the additional requirement that each  $k$ -simplex  $\varphi$  restricts to the identity on  $\Delta^k \times \partial M$ .

The purpose of this section is to prove the following theorem.

**Theorem 5.17.** *Let  $d \geq 3$ . The map*

$$\sigma_*: H_k(B\widetilde{\text{Diff}}_\partial(M_{g,1}); \mathbb{Q}) \rightarrow H_k(B\widetilde{\text{Diff}}_\partial(M_{g+1,1}); \mathbb{Q})$$

*is an isomorphism for  $g > 2k + 4$  and a surjection for  $g = 2k + 4$ .*

We will use the notation

$$\begin{aligned} Y_g &= B\widetilde{\text{Diff}}_\partial(M_{g,1}), \\ X_g &= B\text{aut}_\partial(M_{g,1}). \end{aligned}$$

By Proposition 4.1 we have a commutative diagram with exact rows

$$\begin{array}{ccccccc} 1 & \longrightarrow & \tilde{K}_g & \longrightarrow & \pi_1(Y_g) & \longrightarrow & \Gamma_g \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \parallel \\ 1 & \longrightarrow & K_g & \longrightarrow & \pi_1(X_g) & \longrightarrow & \Gamma_g \longrightarrow 1 \end{array}$$

where  $\Gamma_g$  is the automorphism group of the hyperbolic module  $(H_g, \omega, q)$ , see Example 4.2. The group  $K_g$  is finite. The group  $\tilde{K}_g$  is finite except when  $d \equiv 3 \pmod{4}$ , in which case there is an exact sequence

$$1 \longrightarrow \theta_{2d+1} \longrightarrow \tilde{K}_g \longrightarrow H_g \longrightarrow 1.$$

Let  $J_g$  be the image of  $\pi_1(Y_g) \rightarrow \pi_1(X_g)$ . An easy argument using the above diagram shows that there is a surjective map  $J_g \rightarrow \Gamma_g$  with finite kernel. The

image  $J_g \subset \pi_1(X_g)$  determines a finite cover  $X_g^J \rightarrow X_g$  and there is a homotopy fiber sequence

$$(24) \quad F_g \rightarrow Y_g \rightarrow X_g^J$$

The rational homotopy and homology groups of  $F_g$  were determined in a previous paper using the surgery exact sequence. Let us recall the result.

**Proposition 5.18** (See [9]). *There is an isomorphism of  $\pi_1(X_g^J)$ -modules*

$$(25) \quad H_*(F_g; \mathbb{Q}) \cong \Lambda(\Pi \otimes H_g),$$

*compatible with the stabilization maps. Here,  $\Pi$  is the graded vector space over  $\mathbb{Q}$*

$$\Pi = (\pi_*(G/O) \otimes \mathbb{Q}[-d])^+.$$

*It has basis  $\pi_{\lceil \frac{d+1}{4} \rceil}, \pi_{\lceil \frac{d+1}{4} \rceil + 1}, \dots, \pi_i, \dots$ , with  $\pi_i$  of degree  $4i - d$ . The action of  $\pi_1(X_g^J)$  on the right hand side of (25) is through the standard action of  $\Gamma_g$  on  $H_g$ .*

We may reformulate the previous proposition in terms of polynomial functors.

**Proposition 5.19.** *There is an isomorphism of  $\pi_1(X_g^J)$ -modules*

$$H_*(F_g; \mathbb{Q}) \cong \bigoplus_{k \geq 0} \mathcal{D}(k) \otimes_{\Sigma_k} H_g^{\otimes k}$$

*compatible with the stabilization maps, where  $\mathcal{D}(k)$  is the  $\Sigma_k$ -module*

$$\mathcal{D}(k) = \Pi^{\otimes k}.$$

*In particular,  $\mathcal{D}(k)$  is concentrated in degrees  $\geq k$  for every  $k$ .*

The fibration (24) gives rise to a spectral sequence

$$(26) \quad E_{s,t}^2 = H_s(X_g^J; H_t(F_g; \mathbb{Q})) \Rightarrow H_{s+t}(Y_g; \mathbb{Q})$$

compatible with the stabilization maps.

Since  $X_g^J$  is a cover of  $X_g$ , it has the same universal cover  $\tilde{X}_g$ . The universal cover spectral sequence for  $X_g^J$  with coefficients  $\mathcal{F}_{g,t} = H_t(F_g; \mathbb{Q})$  gives a spectral sequence converging to the  $E^2$ -page of (26).

$$(27) \quad E_{p,q}^2 = H_p(\pi_1(X_g^J); H_q(\tilde{X}_g; \mathcal{F}_{g,t})) \Rightarrow H_{p+q}(X_g^J; \mathcal{F}_{g,t}).$$

By using the two spectral sequences (26) and (27) and the comparison theorem for spectral sequences, the proof of Theorem 5.17 will be completed once we establish that the map

$$\sigma: H_p(\pi_1(X_g^J); H_q(\tilde{X}_g; \mathcal{F}_{g,t})) \rightarrow H_p(\pi_1(X_{g+1}^J); H_q(\tilde{X}_{g+1}; \mathcal{F}_{g+1,t}))$$

is an isomorphism for  $g > 2p + 2q + 2t + 4$  and a surjection for  $g \geq 2p + 2q + 2t + 4$ .

**Proposition 5.20.** *Let  $d \geq 3$  and  $g \geq 2$ . Fix a non-negative integer  $t$  and let  $\mathcal{F}_{g,t} = H_t(F_g; \mathbb{Q})$ , viewed as a  $\pi_1(X_g^J)$ -module. There is an isomorphism of  $\pi_1(X_g^J)$ -modules*

$$H_q(\tilde{X}_g; \mathcal{F}_{g,t}) \cong H_*^{CE}(\mathfrak{g}_g) \otimes \mathcal{F}_{g,t}$$

*that is compatible with the stabilization maps. There results an isomorphism*

$$H_p(\pi_1(X_g^J); H_q(\tilde{X}_g; \mathcal{F}_{g,t})) \cong H_p(\Gamma_g; H_q^{CE}(\mathfrak{g}_g) \otimes \mathcal{F}_{g,t})$$

*compatible with the stabilization maps.*

**Proposition 5.21.** *Let  $d \geq 3$ . The map*

$$\sigma: H_p(\Gamma_g; H_q^{CE}(\mathfrak{g}_g) \otimes \mathcal{F}_{g,t}) \rightarrow H_p(\Gamma_{g+1}; H_q^{CE}(\mathfrak{g}_{g+1}) \otimes \mathcal{F}_{g+1,t})$$

*is an isomorphism for  $g > 2p + 2q + 2t + 4$  and a surjection for  $g \geq 2p + 2q + 2t + 4$ .*

*Proof.* The proof proceeds exactly as the proof of Proposition 5.15, after noting that  $C_*^{CE}(\mathfrak{g}_g) \otimes \mathcal{F}_{g,t}$  is a split complex of  $\mathbb{Q}[\Gamma_g]$ -modules whose module of  $q$ -chains is the value of a polynomial functor of degree  $2q + 2t$  on the standard  $\mathbb{Q}[\Gamma_g]$ -module  $H_g^{\mathbb{Q}}$ . More precisely, by combining Proposition 5.6 and Proposition 5.19 we have that

$$C_*^{CE}(\mathfrak{g}_g) \otimes \mathcal{F}_{g,*} \cong \bigoplus_{k \geq 0} (\mathcal{C} \otimes \mathcal{D})(k) \otimes_{\Sigma_k} H_g^{\otimes k},$$

where

$$(\mathcal{C} \otimes \mathcal{D})(k) = \bigoplus_{i+j=k} \text{Ind}_{\Sigma_i \times \Sigma_j}^{\Sigma_k} \mathcal{C}(i) \otimes \mathcal{D}(j).$$

Since  $\mathcal{C}(i)$  is concentrated in degrees  $\geq \frac{id}{3}$  and  $\mathcal{D}(j)$  is concentrated in degrees  $\geq j$ , it follows that  $(\mathcal{C} \otimes \mathcal{D})(k)$  is concentrated in degrees  $\geq k/2$ . This implies that the functor  $(\mathcal{C} \otimes \mathcal{D})_p(-)$  is polynomial of degree  $\leq 2p$  for every  $p$ .  $\square$

## 6. STABLE COHOMOLOGY

The goal of this section is to prove Theorem 1.2.

### 6.1. The stable cohomology for homotopy automorphisms. Let

$$X_g = B \text{aut}_{\partial}(M_{g,1}),$$

and consider the homotopy colimit

$$X_{\infty} = \text{hocolim}(X_1 \rightarrow X_2 \rightarrow X_3 \rightarrow \cdots),$$

where the homotopy colimit is taken over the stabilization maps described in §5.1. By Theorem 5.12, the canonical map  $X_g \rightarrow X_{\infty}$  induces an isomorphism in rational cohomology

$$H^k(X_{\infty}; \mathbb{Q}) \rightarrow H^k(X_g; \mathbb{Q})$$

for  $g > 2k + 4$ . The goal of this section is to prove the following theorem.

**Theorem 6.1.** *Let  $d \geq 3$ . There is an isomorphism of graded rings*

$$(28) \quad H^*(X_{\infty}; \mathbb{Q}) \cong H^*(\Gamma_{\infty}; \mathbb{Q}) \otimes H_{CE}^*(\mathfrak{g}_{\infty})^{\Gamma_{\infty}}.$$

In the above theorem,  $\Gamma_{\infty} = \text{colim}_g \Gamma_g$  and  $\mathfrak{g}_{\infty} = \text{colim}_g \mathfrak{g}_g$ . Recall that  $\Gamma_g$  denotes the automorphism group of the quadratic module  $(H_g, \mu, q)$  and that  $\mathfrak{g}_g$  denotes the graded Lie algebra  $\text{Der}_{\omega_g}^+(\mathbb{L}(H_g^{\mathbb{Q}}))$ .

The proof has three ingredients. The homological stability theorem together with Theorem 5.11 will allow us to conclude that the universal cover spectral sequence for  $X_{\infty}$  satisfies  $E_2^{p,q} \cong E_2^{p,0} \otimes E_2^{0,q}$ . Then we will prove that the spectral sequence collapses at  $E_2$ . We do this by showing that the rational cohomology ring of  $X_{\infty}$  is free and that the map  $X_{\infty} \rightarrow B\Gamma_{\infty}$  is injective on indecomposables in rational cohomology.

**Theorem 6.2.** *The natural map*

$$H^p(\Gamma_g; \mathbb{Q}) \otimes H_{CE}^q(\mathfrak{g}_g)^{\Gamma_g} \rightarrow H^p(\pi_1(X_g); H^q(\tilde{X}_g; \mathbb{Q})),$$

*is an isomorphism in the stable range  $g > 2p + 2q + 4$ .*

*Proof.* As in Proposition 5.14, there is an isomorphism

$$(29) \quad H^p(\pi_1(X_g); H^q(\tilde{X}_g; \mathbb{Q})) \cong H^p(\Gamma_g; H_{CE}^q(\mathfrak{g}_g)),$$

compatible with the stabilization maps. Since  $\Gamma_g$  is rationally perfect, the functor  $H^p(\Gamma_g; -)$  is exact on the category of finite dimensional  $\mathbb{Q}[\Gamma_g]$ -modules. In particular, since the Chevalley-Eilenberg complex  $C_{CE}^*(\mathfrak{g}_g)$  is finite dimensional in each degree, we may identify the right hand side of (29) with the  $q$ -th cohomology of the cochain complex  $H^p(\Gamma_g; C_{CE}^*(\mathfrak{g}_g))$ . Proposition 5.6 may be used to identify

$C_{CE}^q(\mathfrak{g}_g)$  with the value at  $H_g^{\mathbb{Q}}$  of a polynomial functor of degree  $\leq 2q$ . Hence, by Theorem 5.11 the natural map

$$H^p(\Gamma_g; \mathbb{Q}) \otimes C_{CE}^q(\mathfrak{g}_g)^{\Gamma_g} \rightarrow H^p(\Gamma_g; C_{CE}^q(\mathfrak{g}_g))$$

is an isomorphism for  $g > 2p + 2q + 4$ . The claim follows by passing to cohomology in the  $q$  direction and using (29).  $\square$

**Theorem 6.3.** *The cohomology algebra  $H^*(X_{\infty}; \mathbb{Q})$  is free graded commutative with finitely many generators in each degree.*

*Proof.* It follows from the homological stability theorem that the natural map  $H^k(X_{\infty}; \mathbb{Q}) \rightarrow H^k(X_g; \mathbb{Q})$  is an isomorphism for  $g > 2k + 4$ . The latter group is finite dimensional by Theorem 4.4. This proves the claim about finite type. To show that the cohomology algebra is free, an argument similar to that of [40] will work. Let  $\mathcal{D}_{2d}$  denote the little  $2d$ -disks operad. There are maps

$$(30) \quad \mathcal{D}_{2d}(r) \times X_{g_1} \times \cdots \times X_{g_r} \rightarrow X_g, \quad g = g_1 + \cdots + g_r.$$

Indeed, given a configuration of  $r$  little  $2d$ -disks in a fixed  $2d$ -disk, we may remove their interiors and glue in the manifolds  $M_{g_1,1}, \dots, M_{g_r,1}$  in their place. The result is homeomorphic to  $M_{g,1}$ . Given homotopy automorphisms of  $M_{g_i,1}$  that restrict to the identity on the boundary, we can extend them to a homotopy automorphism of  $M_{g,1}$  by letting it be the identity outside the interiors of the removed disks. This construction respects compositions, so it passes to classifying spaces, giving (30). The maps (30) endow the disjoint union

$$X = \coprod_{g \geq 0} X_g$$

with the structure of an  $E_{2d}$ -space. By the recognition principle for iterated loop spaces (cf. [36, 37]), the space  $X$  admits a group completion  $GX$  which is a  $2d$ -fold loop space, and it follows from the group completion theorem that there is a homology isomorphism  $X_{\infty} \rightarrow GX_0$ , where  $GX_0$  denotes a connected component of  $GX$  (cf. [1, §3.2]). Since  $GX_0$  is a  $2d$ -fold loop space, its rational cohomology is a graded commutative and cocommutative Hopf algebra, so the same is true of the cohomology of  $X_{\infty}$ . By [41], this implies that the cohomology algebra is free graded commutative.  $\square$

**Theorem 6.4.** *The map  $X_{\infty} \rightarrow B\Gamma_{\infty}$  induces an injective homomorphism on indecomposables in rational cohomology.*

Theorem 6.4 will follow from a stronger statement, namely that the composite map

$$B\text{Diff}_{\partial}(M_{\infty,1}) \rightarrow B\text{aut}_{\partial}(M_{\infty,1}) \rightarrow B\Gamma_{\infty}$$

induces an injective map on indecomposables in rational cohomology. This will be proved below, see Theorem 6.7.

*Proof of Theorem 6.1.* We have  $H^1(X_g; \mathbb{Q}) = H^1(B\pi_1(X_g); \mathbb{Q}) = 0$  for  $g \geq 2$ , because the group  $\pi_1(X_g)$  is rationally perfect, see Proposition 5.13. Let  $(X_g)_{\mathbb{Q}}^+$  and  $B\pi_1(X_g)_{\mathbb{Q}}^+$  denote the rational plus constructions (see Appendix C) and let  $T_g$  be the homotopy fiber of the map  $(X_g)_{\mathbb{Q}}^+ \rightarrow B\pi_1(X_g)_{\mathbb{Q}}^+$ . We obtain a map of fibrations,

$$\begin{array}{ccccc} \tilde{X}_g & \longrightarrow & X_g & \longrightarrow & B\pi_1(X_g) \\ \downarrow & & \downarrow & & \downarrow \\ T_g & \longrightarrow & (X_g)_{\mathbb{Q}}^+ & \longrightarrow & B\pi_1(X_g)_{\mathbb{Q}}^+, \end{array}$$

and we may consider the induced map of cohomology spectral sequences  $E \rightarrow \overline{E}$  with  $\mathbb{Q}$ -coefficients;

$$E_2^{p,q} = H^p(B\pi_1(X_g)_\mathbb{Q}^+; H^q(T_g)), \quad \overline{E}_2^{p,q} = H^p(B\pi_1(X_g); H^q(\widetilde{X}_g)).$$

By construction, the maps  $E_2^{p,0} \rightarrow \overline{E}_2^{p,0}$  and  $E_\infty^{p,q} \rightarrow \overline{E}_\infty^{p,q}$  are isomorphisms for all  $p, q$ . We have that  $E_2^{p,q} \cong E_2^{p,0} \otimes E_2^{0,q}$  because the spaces involved are simply connected. In the spectral sequence  $\overline{E}$ , we have cohomology with twisted coefficients, but it follows from Theorem 6.2 that  $\overline{E}_2^{p,q} \cong \overline{E}_2^{p,0} \otimes \overline{E}_2^{0,q}$  for all  $p, q$  in the stable range. The map  $E \rightarrow \overline{E}$  respects these isomorphisms, because they may be realized by taking cup products. Thus, we are in position to apply Zeeman's comparison theorem for spectral sequences; we may conclude that

$$E_2^{0,q} \rightarrow \overline{E}_2^{0,q}$$

is an isomorphism for all  $q$  in the stable range. There results an isomorphism of graded algebras

$$(31) \quad H^*(T_\infty; \mathbb{Q}) \cong H^*(\widetilde{X}_\infty; \mathbb{Q})^{\Gamma_\infty}.$$

The stable cohomology of  $B\pi_1(X_g)_\mathbb{Q}^+$  agrees with the stable rational cohomology of the group  $\Gamma_g$ , because  $\pi_1(X_g)$  surjects onto  $\Gamma_g$  with finite kernel. Borel's calculation of the stable cohomology of arithmetic groups [12] tells us that the cohomology ring  $H^*(\Gamma_\infty; \mathbb{Q})$  is free graded commutative. This, together with Theorem 6.3 and Theorem 6.4, shows that the hypotheses of Lemma 6.5 below are fulfilled, which yields an isomorphism of graded algebras

$$H^*((X_\infty)_\mathbb{Q}^+) = H^*(B\pi_1(X_\infty)_\mathbb{Q}^+) \otimes H^*(T_\infty; \mathbb{Q}).$$

The proof is finished by combining this with the isomorphism (31).  $\square$

For an arbitrary fibration  $F \rightarrow E \rightarrow B$ , injectivity of the map  $H^*(B) \rightarrow H^*(E)$  is not enough to ensure collapse of the associated spectral sequence (see e.g. the discussion in [39, p.148–149]). It is for this reason we need to know that the cohomology ring of  $X_\infty$  is free; we use the following easy lemma. The proof is left to the reader.

**Lemma 6.5.** *Let  $F \rightarrow E \rightarrow B$  be a fibration of simply connected spaces of finite  $\mathbb{Q}$ -type. If the cohomology rings  $H^*(E; \mathbb{Q})$  and  $H^*(B; \mathbb{Q})$  are free graded commutative and if  $H^*(B; \mathbb{Q}) \rightarrow H^*(E; \mathbb{Q})$  is injective on indecomposables, then there is an isomorphism of graded algebras*

$$H^*(E; \mathbb{Q}) \cong H^*(F; \mathbb{Q}) \otimes H^*(B; \mathbb{Q}).$$

**6.2. The stable cohomology of the diffeomorphism group.** The stable cohomology ring of the diffeomorphism group was recently calculated by Galatius and Randal-Williams [24]. Let us recall their description.

Let  $M$  be an oriented  $2d$ -dimensional manifold and let  $\text{Diff}(M)$  be the group of orientation preserving diffeomorphisms of  $M$ . The space  $B\text{Diff}(M)$  is a classifying space for smooth oriented fiber bundles with fiber diffeomorphic to  $M$ .

To every characteristic class of oriented vector bundles

$$c \in H^k(BSO(2d))$$

there is an associated characteristic class of smooth  $M$ -bundles

$$\kappa_c \in H^{k-2d}(B\text{Diff}(M)),$$

constructed as follows. Given a smooth oriented fiber bundle

$$M \rightarrow E \xrightarrow{\pi} X,$$



the vertical tangent bundle  $T_\pi E$  is an oriented  $2d$ -dimensional vector bundle over  $E$ , and we may consider its characteristic class  $c(T_\pi E) \in H^k(E)$ . By applying the Gysin homomorphism  $\pi_! : H^k(E) \rightarrow H^{k-2d}(X)$ , we obtain a class

$$\kappa_c(\pi) := \pi_!(c(T_\pi E)) \in H^{k-2d}(X).$$

This construction is natural with respect to bundle maps, and there is a universal class

$$\kappa_c \in H^{k-2d}(B \operatorname{Diff}(M)).$$

Recall that the rational cohomology of  $BSO(2d)$  is a polynomial ring

$$H^*(BSO(2d); \mathbb{Q}) = \mathbb{Q}[p_1, \dots, p_{d-1}, e]$$

in the Pontryagin classes  $p_i$  and the Euler class  $e$ .

For  $M = M_g$ , the pullback of  $\kappa_c$  along the map  $B \operatorname{Diff}_\partial(M_{g,1}) \rightarrow B \operatorname{Diff}(M_g)$  gives us a class in  $H^{k-2d}(B \operatorname{Diff}_\partial(M_{g,1}))$  that we will also denote  $\kappa_c$ .

**Theorem 6.6** (Galatius-Randal-Williams [24]). *For  $d \neq 2$ , the stable cohomology of the diffeomorphism group of  $M_{g,1}$  is given by*

$$H^*(B \operatorname{Diff}_\partial(M_{\infty,1}); \mathbb{Q}) \cong \mathbb{Q}[\kappa_c | c \in B],$$

where  $B$  is the set of monomials  $c$  in the Pontryagin classes  $p_{d-1}, p_{d-2}, \dots, p_{\lceil \frac{d+1}{4} \rceil}$  and the Euler class  $e$ , of total degree  $|c| > 2d$ .

### 6.3. Borel's calculation of the stable cohomology of arithmetic groups.

We will also need to recall Borel's calculation of the stable cohomology of arithmetic subgroups of Lie groups.

The rational cohomology of  $BU$  is a polynomial algebra in the Chern classes

$$H^*(BU; \mathbb{Q}) = \mathbb{Q}[c_1, c_2, \dots].$$

The Hopf algebra structure is given by  $\Delta(c_n) = \sum_{p+q=n} c_p \otimes c_q$ . Let  $\sigma_1, \dots, \sigma_n$  denote the elementary symmetric polynomials in the indeterminates  $t_1, \dots, t_n$ . Then there is a unique polynomial  $p_n$  such that

$$p_n(\sigma_1, \dots, \sigma_n) = t_1^n + \dots + t_n^n,$$

Define the 'Newton classes'

$$s_n = p_n(c_1, \dots, c_n) \in H^{2n}(BU; \mathbb{Q}).$$

These are primitive generators for the rational cohomology of  $BU$ .

According to Borel [12], the rational cohomology of the infinite symplectic group  $\operatorname{Sp}(\mathbb{Z})$  is the primitively generated Hopf algebra

$$H^*(B \operatorname{Sp}(\mathbb{Z}); \mathbb{Q}) \cong \mathbb{Q}[x_1, x_2, \dots].$$

The primitive generators  $x_i$  are of degree  $4i - 2$ , and may be chosen to be the pullbacks of the odd classes  $s_{2i-1} \in H^{4i-2}(BU; \mathbb{Q})$  along the map

$$B \operatorname{Sp}(\mathbb{Z}) \rightarrow B \operatorname{Sp}(\mathbb{R}) \xleftarrow{\sim} BU.$$

**6.4. Relation between Borel classes and  $\kappa$ -classes.** There is another way of producing characteristic classes of smooth fiber bundles, following Atiyah [4, §4]. Again, let  $M$  be a smooth oriented  $2d$ -dimensional manifold. Assume that  $d$  is odd. Then  $H^d(M; \mathbb{R})$  is of even dimension, say  $2g$ .

Let  $E \xrightarrow{\pi} X$  be an  $M$ -bundle. There is a real  $2g$ -dimensional vector bundle  $\xi$  over  $X$  (the Hodge bundle), with structure group  $\operatorname{Sp}_{2g}(\mathbb{R})$ , whose fiber over  $x$  is the cohomology group

$$\xi_x = H^d(\pi^{-1}(x); \mathbb{R}).$$

The structure group can be reduced to the maximal compact subgroup  $U(g) \subset \mathrm{Sp}_{2g}(\mathbb{R})$ , so we obtain a  $g$ -dimensional complex vector bundle  $\eta$  over  $X$ . We may consider the ‘Newton classes’

$$s_i(\eta) \in H^{2i}(X).$$

The even classes vanish,  $s_{2i}(\eta) = 0$ . The odd classes agree with the pullbacks of the Borel classes

$$s_{2i-1} = \rho^*(x_i)$$

along the map  $\rho: X \rightarrow B\mathrm{Sp}_{2g}(\mathbb{Z})$ .

Now, one may ask if there are any relations between the classes  $\kappa_s(T_\pi E)$  and  $s_i(\eta)$ , where  $T_\pi E$  is the vertical tangent bundle. This problem was addressed and solved by Morita [43, §2] in the case of surface bundles. A similar treatment is possible in our situation. According to [4, (4.3)], we have the relation

$$(32) \quad ch(\eta^* - \eta) = \pi_!(\tilde{L}(T_\pi E))$$

in the cohomology of  $X$ . In the left hand side,  $\eta^*$  denotes the conjugate bundle, the formal difference  $\eta^* - \eta$  is taken in  $K(X)$ , and  $ch$  is the Chern character  $ch: K(X) \rightarrow H^*(X; \mathbb{Q})$ ,

$$ch(\eta) = g + \sum_{k \geq 1} \frac{s_k(\eta)}{k!}.$$

Since  $s_k(\eta^*) = (-1)^k s_k(\eta)$ , we may write the left hand side of (32) as

$$ch(\eta^* - \eta) = -2 \sum_{k \text{ odd}} \frac{s_k(\eta)}{k!}.$$

Turning to the right hand side, if  $\xi$  is a real vector bundle over  $E$  of dimension  $2d$ , then

$$\tilde{L}(\xi) = \tilde{L}(p_1, \dots, p_d)$$

is the formal power series in the Pontryagin classes of  $\xi$  determined by

$$\tilde{L}(\sigma_1, \dots, \sigma_d) = f(t_1) \cdots f(t_d),$$

where  $\sigma_i$  is the elementary symmetric polynomial in  $t_1^2, \dots, t_d^2$  of degree  $i$ , and  $f(t)$  is the formal power series

$$f(t) = \frac{t}{\tanh(t/2)} = 2 \left( 1 + \sum_{k \geq 1} (-1)^{k-1} \frac{B_k}{(2k)!} t^{2k} \right).$$

Here  $B_k$  are the Bernoulli numbers. Explicitly, the homogeneous term in  $\tilde{L}(\xi)$  of degree  $n$  is given by

$$\tilde{L}_n(p_1, \dots, p_n) = 2^d \sum_{I \vdash n} \lambda_I s_I(p_1, \dots, p_n).$$

The sum is over all partitions  $I = (i_1, \dots, i_r)$  of  $n$ , and  $s_I$  denotes the corresponding polynomial in the elementary symmetric polynomials (see e.g. [42, p.188]). The coefficients are  $\lambda_I = \lambda_{i_1} \cdots \lambda_{i_r}$ , where

$$\lambda_k = (-1)^{k-1} \frac{B_k}{(2k)!}.$$

For a partition  $I \vdash n$ , let

$$\kappa_I(\pi) := \pi_!(s_I(p_1, \dots, p_n)) \in H^{4n-2d}(X; \mathbb{Q}).$$

We are assuming that  $d$  is odd, say  $d = 2s + 1$ . Then, by comparing homogeneous terms in (32), we obtain the relation

$$(33) \quad -2 \frac{s_{2i-1}(\eta)}{(2i-1)!} = 2^d \sum_{I \vdash i+s} \lambda_I \kappa_I(\pi) \in H^{4i-2}(X; \mathbb{Q})$$

for every  $i$ .

Passing to the universal bundle, we may conclude that the image of each Borel class  $x_i \in H^{4i-2}(B\mathrm{Sp}(\mathbb{Z}); \mathbb{Q})$  under the map

$$(34) \quad H^*(B\mathrm{Sp}(\mathbb{Z}); \mathbb{Q}) \rightarrow H^*(B\mathrm{Diff}_\partial(M_{\infty,1}); \mathbb{Q})$$

is a non-zero linear combination of  $\kappa$ -classes. In particular, the map (34) is injective on indecomposables. Thus, we have proved the following theorem, for  $d$  odd. A similar argument may be carried out in the case when  $d$  is even.

**Theorem 6.7.** *Let  $d \geq 3$ . The map  $B\mathrm{Diff}_\partial(M_{\infty,1}) \rightarrow B\Gamma_\infty$  is injective on indecomposables in rational cohomology.*

## 7. KONTSEVICH'S THEOREM AND OUTER AUTOMORPHISM GROUPS OF FREE GROUPS

The results of the previous section yield the following description of the stable cohomology of the homotopy automorphisms:

$$(35) \quad H^*(B\mathrm{aut}_\partial(M_{\infty,1}); \mathbb{Q}) \cong H^*(\Gamma_\infty; \mathbb{Q}) \otimes H_{CE}^*(\mathfrak{g}_\infty)^{\Gamma_\infty}.$$

Borel's calculation [12] allows us to identify the left tensor factor with a polynomial algebra

$$H^*(\Gamma_\infty; \mathbb{Q}) \cong \begin{cases} \mathbb{Q}[x_1, x_2, \dots], & |x_i| = 4i - 2, \quad d \text{ odd}, \\ \mathbb{Q}[x_1, x_2, \dots], & |x_i| = 4i, \quad d \text{ even}. \end{cases}$$

For  $d$  odd, something very close to the right factor in (35) has been considered by Kontsevich [28, 29], but for different purposes. In essence, Kontsevich shows that the invariants of the Lie algebra cohomology can be calculated in terms of graph homology. Via a theorem of Culler-Vogtmann [20], he is able to relate his graph homology to the homology of outer automorphism groups of free groups. We refer the reader to [18] for a detailed proof of Kontsevich's theorem. After making the necessary modifications, Kontsevich's theorem yields the following description.

**Theorem 7.1** (Kontsevich). *For  $d$  odd, the ring of invariants,*

$$H_{CE}^*(\mathfrak{g}_\infty)^{\Gamma_\infty},$$

*is isomorphic to the free graded commutative algebra on a graded vector space  $V$ , whose graded components are given by*

$$V^k = \bigoplus_{n>0} H_{2nd-k}(\mathrm{Out} F_{n+1}; \mathbb{Q}).$$

*Here,  $\mathrm{Out} F_{n+1}$  denotes the outer automorphism group of the free group on  $n+1$  generators.*

Our description of the cohomology of  $B\mathrm{aut}_\partial(M_{\infty,1})$  in Theorem 6.1 combined with Kontsevich's theorem has the following remarkable consequence. To every homology class

$$c \in H_k(\mathrm{Out} F_{n+1}; \mathbb{Q})$$

there is an associated characteristic class of fibrations

$$\lambda_c \in H^{2nd-k}(B\mathrm{aut}_\partial(M_{\infty,1}); \mathbb{Q}).$$

Moreover, linearly independent classes  $c$  give algebraically independent classes  $\lambda_c$ . It is amusing to compare this with the description of the stable cohomology of the

diffeomorphism group in terms of  $\kappa$ -classes, see Theorem 6.6. However, the situation for homotopy automorphisms is not as well understood at the moment. First of all, unlike for  $BSO(n)$ , the rational homology of the outer automorphism group  $\text{Out } F_{n+1}$  is far from understood, although it has been the subject of a lot of recent research, see [19, 44]. Secondly, the geometric meaning of the ‘ $\lambda$ -classes’ is somewhat convoluted. One naturally wonders whether there is a geometric construction of the characteristic classes  $\lambda_c$  that does not depend on Kontsevich’s theorem. These difficulties notwithstanding, we arrive at the following description.

**Theorem 7.2.** *Let  $d \geq 3$  be odd. For every  $n \geq 2$  and every  $k \geq 0$ , let  $B_{n,k}$  be a basis for the vector space  $H_k(\text{Out } F_{n+1}; \mathbb{Q})$ . The rational stable cohomology ring*

$$H^*(B \text{aut}_\partial(M_{\infty,1}); \mathbb{Q})$$

*is a free graded commutative algebra on the ‘Borel classes’,*

$$x_1, x_2, \dots, \text{ of degree } |x_i| = 4i - 2,$$

*and the ‘ $\lambda$ -classes’,*

$$\lambda_c \text{ for every } c \in B_{n,k}, \text{ of degree } |\lambda_c| = 2nd - k.$$

It is possible to draw some immediate interesting conclusions from this description.

**Corollary 7.3.** *For  $d \geq 3$  odd, the map in cohomology*

$$(36) \quad H^*(B \text{aut}_\partial(M_{\infty,1}); \mathbb{Q}) \rightarrow H^*(B \text{Diff}_\partial(M_{\infty,1}); \mathbb{Q})$$

*is non-trivial, but neither surjective nor injective.*

*Proof.* Euler characteristic calculations have recently shown the existence of non-trivial odd dimensional homology classes for  $\text{Out } F_n$ , see [44]. By Theorem 7.2, this implies that there are odd dimensional algebra generators for  $H^*(B \text{aut}_\partial(M_{\infty,1}); \mathbb{Q})$ . On the other hand, by Theorem 6.6, the cohomology  $H^*(B \text{Diff}_\partial(M_{\infty,1}); \mathbb{Q})$  is concentrated in even dimensions. So the map (36) cannot be injective. By listing known classes in low dimensions, it is easy to see that the map cannot be surjective. On the other hand, the map is non-trivial; at least the Borel classes have non-zero images, cf. Theorem 6.7.  $\square$

This is in sharp contrast with what happens for  $d = 1$ ; for  $S_{g,1}$  an orientable genus  $g$  surface with one boundary component, the inclusion of  $\text{Diff}_\partial(S_{g,1})$  into  $\text{aut}_\partial(S_{g,1})$  is a homotopy equivalence if  $g > 1$ . In particular, the map

$$H^*(B \text{aut}_\partial(S_{\infty,1}); \mathbb{Q}) \rightarrow H^*(B \text{Diff}_\partial(S_{\infty,1}); \mathbb{Q})$$

is an isomorphism.

It would be interesting to get a better understanding of the map (36). It factors through the stable cohomology ring of the block diffeomorphism group, so it would be desirable to get a description of this ring. Another natural question is what happens when  $d$  is even. In this case, Kontsevich’s theorem does not provide any information. We hope to address these problems in a sequel.

## APPENDIX A. COHOMOLOGY OF ARITHMETIC GROUPS

The automorphism groups  $\text{Aut}(H, \mu, q)$  and  $\text{Aut}(H, \mu, Jq)$  associated to a quadratic module  $(H, \mu, q)$  are arithmetic. We will summarize the results on the cohomology of arithmetic groups that we need. We refer to Serre’s survey article [46], and the references therein, for more details.

**Theorem A.1.** *Let  $G$  be an algebraic group defined over  $\mathbb{Q}$ , let  $\Gamma$  be an arithmetic subgroup of  $G_{\mathbb{Q}}$ , and let  $V$  be a finite dimensional  $\mathbb{Q}$ -vector space with an action of  $\Gamma$ . Then*

- (1) *The cohomology groups  $H^k(\Gamma; V)$  are finite dimensional.*
- (2) *If  $G$  is simple and of  $\mathbb{Q}$ -rank at least 2, then the first cohomology group vanishes,  $H^1(\Gamma; V) = 0$ .*

*Proof.* If  $\Gamma$  is torsion-free the first claim follows from the fact that the trivial  $\mathbb{Z}[\Gamma]$ -module  $\mathbb{Z}$  admits a finite length resolution by finitely generated free  $\mathbb{Z}[\Gamma]$ -modules. For general  $\Gamma$ , there exists a torsion-free subgroup  $\Gamma' \subseteq \Gamma$  of finite index, and the claim follows because  $H^k(\Gamma; V)$  may be identified with the set of  $\Gamma$ -invariants in  $H^k(\Gamma'; V)$  by a transfer argument (see e.g., [16, III.(10.4)]).

If  $G$  is simple and of  $\mathbb{Q}$ -rank at least 2, every finite dimensional representation  $V$  of  $\Gamma$  is almost algebraic (see [46, 1.3(9)]). This means that there is a finite index subgroup  $\Gamma' \subseteq \Gamma$  such that the restriction of  $V$  to  $\Gamma'$  is the restriction of a rational representation of the algebraic group  $G$ . This implies that the first cohomology group  $H^1(\Gamma; V)$  vanishes.  $\square$

## APPENDIX B. SOME ELEMENTARY HOMOLOGICAL ALGEBRA

We will consider  $\mathbb{Z}$ -graded chain complexes over an associative ring  $R$ , e.g.,  $R = \mathbb{Q}[\pi]$  for a group  $\pi$ .

A chain complex  $C_*$  is called split if there are maps  $s_n: C_n \rightarrow C_{n+1}$  such that  $dsd = d$ . Equivalently, there is a chain homotopy equivalence between  $C_*$  and the homology  $H_*(C)$ , viewed as a chain complex with trivial differential.

**Lemma B.1.** *If  $C_*$  is a split chain complex then there is a chain homotopy equivalence*

$$C_* \xrightarrow[p_C]{\simeq} H_*(C)$$

*such that  $p_C(z) = [z]$  if  $z$  is a cycle.*

*If  $f: C_* \rightarrow D_*$  is a chain map between split chain complexes (not necessarily compatible with the splittings), then the diagram*

$$\begin{array}{ccc} C_* & \xrightarrow{f} & D_* \\ p_C \downarrow & & \downarrow p_D \\ H_*(C) & \xrightarrow{H_*(f)} & H_*(D) \end{array}$$

*commutes up to chain homotopy.*

*Proof.* Let  $s: C_* \rightarrow C_{*+1}$  satisfy  $dsd = d$ . The reader may check that the formulas

$$h \circ C_* \xrightarrow[p]{\simeq} H_*(C)$$

$$p(x) = [x - sd(x)], \quad \nabla[z] = z - ds(z), \quad h = s - s^2d$$

give well-defined maps that satisfy

$$p\nabla = 1, \quad 1 - \nabla p = dh + hd.$$

Clearly,  $p(z) = [z]$  if  $z$  is a cycle.

Next, consider a chain map  $f: C_* \rightarrow D_*$  between split chain complexes. Since  $p_C(z) = [z]$  for cycles  $z$ , we have that  $\nabla_C[z] = z - dh(z)$ . Therefore,

$$\begin{aligned} p_D f \nabla_C [z] &= p_D f (z - dh(z)) \\ &= p_D (f(z)) \\ &= [f(z)], \end{aligned}$$

showing that  $p_D f \nabla_C = H_*(f)$ . Hence,

$$H_*(f)p_C = p_D f \nabla_C p_C \simeq p_D f.$$

□

**Lemma B.2.** *A chain complex  $C_*$  is split if and only if the short exact sequences*

$$\begin{aligned} 0 \rightarrow Z_n \rightarrow C_n \xrightarrow{d_n} B_{n-1} \rightarrow 0 \\ 0 \rightarrow B_n \rightarrow Z_n \rightarrow H_n \rightarrow 0 \end{aligned}$$

*are split exact for all  $n$ . Here,  $Z_n = \ker(d_n)$ ,  $B_{n-1} = \text{im}(d_n)$ , and  $H_n = H_n(C_*)$ .*

**Definition B.3.** We will say that a group  $\pi$  is *rationally perfect* if  $H^1(\pi; V) = 0$  for all finite dimensional vector spaces  $V$  over  $\mathbb{Q}$  with an action of  $\pi$ .

**Lemma B.4.** *A group  $\pi$  is rationally perfect if and only if  $\text{Ext}_{\mathbb{Q}[\pi]}^1(W, V) = 0$  for all finite dimensional vector spaces  $V$  and  $W$  over  $\mathbb{Q}$  with an action of  $\pi$ .*

*Proof.* Use the relation  $\text{Ext}_{\mathbb{Q}[\pi]}^1(W, V) \cong H^1(\pi; \text{Hom}_{\mathbb{Q}}(W, V))$ . □

**Proposition B.5.** *Let  $\pi$  be a rationally perfect group. If  $C_*$  is a chain complex of  $\mathbb{Q}[\pi]$ -module such that  $C_n$  is finite dimensional over  $\mathbb{Q}$  for every  $n$ , then  $C_*$  is split.*

*Proof.* If  $C_n$  is finite dimensional over  $\mathbb{Q}$  for all  $n$ , then so are  $Z_n$ ,  $B_n$  and  $H_n$ . Since  $\pi$  is rationally perfect, the Ext-groups  $\text{Ext}_{\mathbb{Q}[\pi]}^1(H_n, B_n)$  and  $\text{Ext}_{\mathbb{Q}[\pi]}^1(B_{n-1}, Z_n)$  vanish for all  $n$ , which forces  $C_*$  to split by Lemma B.2. □

## APPENDIX C. A $\mathbb{Q}$ -LOCAL PLUS CONSTRUCTION

Let  $X$  be a connected space of finite  $\mathbb{Q}$ -type such that  $H_1(X; \mathbb{Q}) = 0$ . Then  $X$  admits a minimal Sullivan model  $\mathcal{M}_X$  of finite type with generators in degree 2 and above, see e.g., [23, Proposition 12.2]. The spatial realization  $|\mathcal{M}_X|$  is then a simply connected  $\mathbb{Q}$ -local space of finite  $\mathbb{Q}$ -type. Moreover, the canonical map  $X \rightarrow |\mathcal{M}_X|$  is a rational cohomology isomorphism. One may view  $|\mathcal{M}_X|$  as a  $\mathbb{Q}$ -local version of the plus construction, and we will denote it by  $X_{\mathbb{Q}}^+$ . In fact, if the fundamental group of  $X$  is perfect, i.e.,  $H_1(X; \mathbb{Z}) = 0$ , then  $X_{\mathbb{Q}}^+$  is a  $\mathbb{Q}$ -localization of the ordinary plus construction. Note however that the rational plus construction has a wider range of applicability as it does not require the fundamental group to be perfect.

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